THE DE RHAM-HODGE-SKRYPNIK THEORY OF DELSARTE TRANSMUTATION OPERATORS IN MULTIDIMENSION AND ITS APPLICATIONS. PART 1

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Abstract. Spectral properties od Delsarte transmutation operators are studied, their differential geometrical and topological structure in multidimension is analyzed, the relationships with De Rham-Hodge-Skrypnik theory of generalized differential complexes is stated.

1. Spectral operators and generalized eigenfunctions expansions

1.1. Let $\mathcal{H}$ be a Hilbert space in which there is defined a linear closable operator $L \in \mathcal{L}(\mathcal{H})$ with a dense domain $D(L) \subset \mathcal{H}$. Consider the standard quasi-nucleous Gelfand rigging [8] of this Hilbert space $\mathcal{H}$ with corresponding positive $\mathcal{H}_+$ and negative $\mathcal{H}_-$ Hilbert spaces as follows:

$$D(L) \subset \mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_- \subset D'(L), \tag{1.1}$$

being suitable for proper analyzing the spectral properties of the operator L in $\mathcal{H}$. We shall use below the following definition motivated by considerations from [8], Chapter 5.

Definition 1.1. An operator $L \in \mathcal{L}(\mathcal{H})$ will be called **spectral** if for all Borel subsets $\Delta \subset \sigma(L)$ of the spectrum $\sigma(L) \subset \mathbb{C}$ and for all pairs $(u, v) \in \mathcal{H}_+ \times \mathcal{H}_+$ there are defined the following expressions:

$$L = \int_{\sigma(L)} \lambda \mathrm{dE}(\lambda), \quad (u, \mathrm{E}(\Delta)v) = \int_{\Delta} (u, \mathrm{P}(\lambda)v)\mathrm{d}\rho_\sigma(\lambda), \tag{1.2}$$

where ρ_σ is some finite Borel measure on the spectrum $\sigma(L)$, E is some self-adjoint projection operator measure on the spectrum $\sigma(L)$, such that $\mathrm{E}(\Delta)\mathrm{E}(\Delta') = \mathrm{E}(\Delta \cap \Delta')$ for any Borel subsets $\Delta, \Delta' \subset \sigma(L)$, and $\mathrm{P}(\lambda) : \mathcal{H}_+ \to \mathcal{H}_-$, $\lambda \in \sigma(L)$, is the corresponding family of nucleous integral operators from $\mathcal{H}_+$ into $\mathcal{H}_-$.

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As a consequence of the expression (1.2) one can write down that formally in the weak topology of $\mathcal{H}$

$$\mathrm{E}(\Delta)=\int_{\Delta}\mathrm{P}(\lambda)\mathrm{d}\rho_{\sigma}(\lambda) \tag{1.3}$$

for any Borel subset $\Delta\subset\sigma(L)$.

Similarly to (1.2) and (1.3) can write down the corresponding expressions for the adjoint spectral operator $L^{*}\in\mathcal{L}(\mathcal{H})$ whose domain $\mathrm{D}(L^{*})\subset\mathcal{H}$ is assumed to be also dense in $\mathcal{H}$:

$$(\mathrm{E}^{*}(\Delta)u,v)=\int_{\Delta}(P^{*}(\lambda)u,v)\mathrm{d}\rho_{\sigma}^{*}(\lambda), \tag{1.4}$$

$$\mathrm{E}^{*}(\Delta)=\int_{\Delta}P^{*}(\lambda)\mathrm{d}\rho_{\sigma}^{*}(\lambda),$$

where E^{*} is the corresponding projection spectral measure on Borel subsets $\Delta\in\sigma(L^{*})$, $\mathrm{P}^{*}(\lambda):\mathcal{H}\ \rightarrow\ \mathcal{H}$, $\lambda\in\sigma(L^{*})$, is the corresponding family of nucleous integral operators in $\mathcal{H}$ and ρ_{σ}^{*} is some finite Borel measure on the spectrum $\sigma(L^{*})$. We will assume, moreover, that the following conditions

$$\mathrm{P}(\mu)(L-\mu\mathrm{I})v=0,\quad \mathrm{P}^{*}(\lambda)(L^{*}-\bar{\lambda}\mathrm{I})u=0 \tag{1.5}$$

hold for all $u\in\mathrm{D}(L^{*})$, $v\in\mathrm{D}(L)$, where $\bar{\lambda}\in\sigma(L^{*}),\ \mu\in\sigma(L)$. In particular, one assumes also that $\sigma(L^{*})=\bar{\sigma}(L)$.

1.2. Proceed now to a description of the corresponding to operators L and L^{*} generalized eigenfunctions via the approach devised in [8]. We shall speak that an operator $L\in\mathcal{L}(\mathcal{H})$ with a dense domain $\mathrm{D}(L)$ allows a rigging continuation, if one can find another dense in $\mathcal{H}_{+}$ topological subspace $\mathrm{D}_{+}(L^{*})\subset\mathrm{D}(L^{*})$, such that the adjoint operator $L^{*}\in\mathcal{L}(\mathcal{H})$ maps it continuously into $\mathcal{H}_{+}$.

Definition 1.2. A vector $\psi_{\lambda}\in\mathcal{H}_{-}$ is called a generalized eigenfunction of the operator $L\in\mathcal{L}(\mathcal{H})$ corresponding to an eigenvalue $\lambda\in\sigma(L)$ if

$$((L^{*}-\bar{\lambda}\mathrm{I})u,\psi_{\lambda})=0 \tag{1.6}$$

for all $u\in\mathrm{D}_{+}(L^{*})$.

It is evident that in the case when $\psi_{\lambda}\in\mathrm{D}(L),\quad \lambda\in\sigma(L)$, then $L\psi_{\lambda}=\lambda\psi_{\lambda}$ as usually. The definition (1.6) is related [8] with some extension of the operator $L:\mathcal{H}\ \mapsto\ \mathcal{H}$. Since the operator $L^{*}:\ \mathrm{D}_{+}(L^{*})\ \rightarrow\mathcal{H}_{+}$ is continuous one can define the adjoint operator $L_{ext}:=\ L^{*,+}:\ \mathcal{H}_{-}\ \rightarrow\ \mathrm{D}(L^{*})$ with respect to the standard scalar product in $\mathcal{H}$, that is

$$(L^{*}v,u)=(v,L^{*,+}u) \tag{1.7}$$

for any $v\in\mathrm{D}_{+}(L^{*})$ and $u\in\mathcal{H}_{-}$ and coinciding with the operator $L:\mathcal{H}\ \rightarrow\ \mathcal{H}$ upon $\mathrm{D}(L)$. Now the definition (1.6) of a generalized eigenfunction $\psi_{\lambda}\in\mathcal{H}_{-}$ for $\lambda\in\sigma(L)$ is equivalent to the standard expression

$$L_{ext}\psi_{\lambda}=\lambda\psi_{\lambda}. \tag{1.8}$$

If to define the scalar product

$$(u,v):=\ (u,v)_{+}+(L^{*}u,L^{*}v)_{+} \tag{1.9}$$

on the dense subspace $\mathrm{D}_{+}(L^{*})\subset\mathcal{H}_{+}$, then this subspace can be transformed naturally into the Hilbert space $\mathrm{D}_{+}(L^{*})$, whose adjoint "negative" space $\mathrm{D}_{+}^{\prime}(L^{*}):=$

$D_-(L^*) \supset \mathcal{H}_-$. Take now any generalized eigenfunction $\psi_\lambda \in Im\ P(\lambda) \subset \mathcal{H}_-,\ \lambda \in \sigma(L)$, of the operator $L : \mathcal{H} \rightarrow \mathcal{H}$. Then, as one can see from (1.5), $L^*_{ext}\varphi_\lambda = \bar{\lambda}\varphi_\lambda$ for some function $\varphi_\lambda \in Im\ P^*(\lambda) \subset \mathcal{H}_-,\ \bar{\lambda} \in \sigma(L^*)$, and $L^*_{ext} : \mathcal{H}_- \rightarrow D_-(L)$ is the corresponding extension of the adjoint operator $L^* : \mathcal{H} \rightarrow \mathcal{H}$ by means of reducing, as above, the domain $D(L)$ to a new dense in $\mathcal{H}_+$ domain $D_+(L) \subset D(L)$ on which the operator $L : D_+(L) \rightarrow \mathcal{H}_+$ is continuous.

2. Semi-linear forms, generalized kernels and congruence of operators

2.1. Let us consider any continuous semi-linear form $K : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{C}$ in a Hilbert space $\mathcal{H}$. The following classical theorem holds.

Theorem 2.1. *(**L. Schwartz**; see [8]) Consider a standard Gelfand rigged chain of Hilbert spaces (1.1) which is, as usually, invariant under the complex involution $\mathbb{C} :\rightarrow \mathbb{C}^*$. Then any continuous semi-linear form $K : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{C}$ can be written down by means of a generalized kernel $\hat{K} \in \mathcal{H}_- \otimes \mathcal{H}_-$ as follows:*

$$K[u, v] = (\hat{K}, v \otimes u)_{\mathcal{H}\times\mathcal{H}} \tag{2.1}$$

for any $u, v \in \mathcal{H}_+ \subset \mathcal{H}$. The kernel $\hat{K} \in \mathcal{H}_- \otimes \mathcal{H}_-$ allows the representation

$$\hat{K} = (D \otimes D)\bar{K},$$

where $\bar{K} \in \mathcal{H} \otimes \mathcal{H}$ is a usual kernel and $D : \mathcal{H} \rightarrow \mathcal{H}_-$ is the square root $\sqrt{J^*}$ from a positive operator $J^* : \mathcal{H} \rightarrow \mathcal{H}_-$, being a Hilbert-Schmidt embedding of $\mathcal{H}_+$ into $\mathcal{H}$ with respect to the chain (1.1). Moreover, the related kernels $(D \otimes I)\bar{K}$, $(I \otimes D)\bar{K} \in \mathcal{H} \times \mathcal{H}$ are usual ones too.

Take now, as before, an operator $L : \mathcal{H} \rightarrow \mathcal{H}$ with a dense domain $D(L) \subset \mathcal{H}$ allowing the Gelfand rigging continuation (1.1) introduced in the preceding chapter. Denote also by $D_+(L^*) \subset D(L^*)$ the related dense in $\mathcal{H}_+$ subspace.

Definition 2.2. A set of generalized kernels $\hat{Z}_\lambda \subset \mathcal{H}_- \otimes \mathcal{H}_-$ for $\lambda \in \sigma(L) \cap \bar{\sigma}(L^*)$ will be called **elementary** concerning the operator $L : \mathcal{H} \rightarrow \mathcal{H}$ if for any $\lambda \in \sigma(L) \cap \bar{\sigma}(L^*)$, the norm $||\hat{Z}_\lambda||_{\mathcal{H}_-\otimes\mathcal{H}_-} < \infty$ and

$$(\hat{Z}_\lambda, ((\Delta - \lambda I)v) \otimes u) = 0, \quad (\hat{Z}_\lambda, v \otimes (\mathbb{L}^* - \lambda\ I)u) = 0 \tag{2.2}$$

for all $(u, v) \in \mathcal{H}_- \otimes \mathcal{H}_-$.

2.2. Assume further, as above, that all our functional spaces are invariant with respect to the involution $\mathbb{C} :\rightarrow \mathbb{C}^*$ and put $D_+ := D_+(L^*) = D_+(L) \subset \mathcal{H}_+$. Then one can build the corresponding extensions $L_{ext} \supset L$ and $L^*_{ext} \supset L^*$, being linear operators continuously acting from $\mathcal{H}_-$ into $\mathcal{D}_- := D'_+$. The chain (1.1) is now extended to the chain

$$D_+ \subset \mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_- \subset D_- \tag{2.3}$$

and is assumed also that the unity operator $I : \mathcal{H}_- \rightarrow \mathcal{H}_- \subset D_-$ is extended naturally as the imbedding operator from $\mathcal{H}_-$ into D_-. Then equalities (2.2) can be equivalently written down [8] as follows:

$$(L_{ext} \otimes I)\ \hat{Z}_\lambda = \lambda\hat{Z}_\lambda, \quad (I \otimes L^*_{ext})\ \hat{Z}_\lambda = \lambda\hat{Z}_\lambda \tag{2.4}$$

for any $\lambda \in \sigma(L)\cap\bar{\sigma}(L^*)$. Take now a kernel $\hat{\mathrm{K}}_\lambda \in \mathcal{H}_-\otimes\mathcal{H}_-$ and suppose that the following operator equality

$$(L_{ext}\otimes \mathrm{I})\ \hat{\mathrm{K}}_\lambda = (\mathrm{I}\otimes L^*_{ext})\ \hat{\mathrm{K}}_\lambda \tag{2.5}$$

holds. Since the equation (2.4) can be written down in the form

$$(L_{ext}\otimes \mathrm{I})\ \hat{\mathrm{Z}}_\lambda = (\mathrm{I}\otimes L^*_{ext})\ \hat{\mathrm{Z}}_\lambda \tag{2.6}$$

for any $\lambda \in \sigma(L)\cap\bar{\sigma}(L^*)$, the following characteristic theorem [8] holds.

Theorem 2.3. *(see* [8]*, chapter 8, p.621) Let a kernel* $\hat{\mathrm{K}} \in \mathcal{H}_-\otimes\mathcal{H}_-$ *satisfy the condition (2.5). Then due to (2.6) there exists such a finite Borel measure defined on Borel subsets* $\Delta\subset\sigma(L)\cap\bar{\sigma}(L^*)$*, that the following weak spectral representation*

$$\hat{\mathrm{K}} = \int_{\sigma(L)\cap\bar{\sigma}(L^*)} \hat{\mathrm{Z}}_\lambda \mathrm{d}\rho_\sigma(\lambda) \tag{2.7}$$

holds. Moreover, due to (2.4) one can write down the following representation

$$\hat{\mathrm{Z}}_\lambda = \psi_\lambda\otimes\varphi_\lambda,$$

where $L_{ext}\ \psi_\lambda = \lambda\psi_\lambda,\ \ L^*_{ext}\ \varphi_\lambda = \bar{\lambda}\varphi_\lambda,\ \ (\psi_\lambda,\varphi_\lambda)\in\mathcal{H}_-\otimes\mathcal{H}_-$ *and* $\lambda\in\sigma(L)\cap\bar{\sigma}(L^*)$.

Proof. $\triangleleft$ It is easy to see due to (2.6) that the kernel (2.7) satisfies the equation (2.5). On one hand-side, consider the second expression of (1.2) related with our operator $L\in\mathcal{L}(\mathcal{H})$ and observe that it can be represented exactly in the form (2.1):

$$(u,\mathrm{P}(\lambda)v) = (\hat{\mathrm{Z}}_\lambda, v\otimes u)_+ \tag{2.8}$$

for any $u,v\in\mathcal{H}_+$ due to the Schwartz theorem 2.1. On another hand-side due to the condition (2.5) by means of the kernel $\widehat{\mathrm{K}}\in\mathcal{H}_-\otimes\mathcal{H}_-$ one can define as in [8] a new Hilbert space $\mathcal{H}_K\supset\mathcal{H}_+$ with the scalar product

$$(u,v)_{\mathrm{K}} := (|\hat{\mathrm{K}}|, v\otimes u)_{\mathcal{H}\times\mathcal{H}} \tag{2.9}$$

for any $u,v\in\mathcal{H}_+$, where, by definition, $|\hat{\mathrm{K}}|{:=}\sqrt{\hat{\mathrm{K}}^*\hat{\mathrm{K}}}$. As the norm

$$||u||_{\mathrm{K}} = (|\hat{\mathrm{K}}|, u\otimes u)_{\mathcal{H}\times\mathcal{H}} \le ||\hat{\mathrm{K}}||_-\cdot||u\otimes u||_+ = ||\hat{\mathrm{K}}||_-||u||_+^2 \tag{2.10}$$

for any $u\in\mathcal{H}_+$, then one can deduce from (2.10) that really $\mathcal{H}_K\supset\mathcal{H}_+$. Thus one has a new Hilbert-Schmidt rigged chain with the basic Hilbert space taken now to be $\mathcal{H}_{\mathrm{K}}$:

$$\mathcal{H}_{++}\subset(\mathcal{H}_+)\subset\mathcal{H}_{\mathrm{K}}\subset\mathcal{H}_{-\,-}, \tag{2.11}$$

where embeddings $\mathcal{H}_{++}\ \to\ \mathcal{H}_{\mathrm{K}}$ is also quasi-nucleous [8] as the composition of the quasi-nucleous imbedding $\mathcal{H}_{++}\ \to\ \mathcal{H}_+$ and the continuous imbedding $\mathcal{H}_+\to\mathcal{H}_{\mathrm{K}}$ due to (2.10). Since now the operator $L\in\mathcal{L}(\mathcal{H})$ can be considered as an operator $L\in\mathcal{L}(\mathcal{H}_{\mathrm{K}})$, there exists a representation similar to (2.8) but just for $u,v\in\mathcal{H}_{++}$. Thereby for the expression (2.9) one derives from (2.4) the searched for all $(u,v)\in\mathcal{H}_+\times\mathcal{H}_+$ representation :

$$(\hat{\mathrm{K}}, v\otimes u)_{\mathcal{H}\times\mathcal{H}} = (u,v)_{\mathrm{K}} = \int_{\sigma(L)\cap\bar{\sigma}(L^*)} (\hat{\mathrm{Z}}_\lambda, v\otimes u)_{\mathrm{K}}\mathrm{d}\rho_\sigma(\lambda), \tag{2.12}$$

equivalent, obviously, to (2.7). The integral (2.7) is defined well since the norm $||\widehat{\mathrm{Z}}_\lambda||_{-\,-}<\infty$ and the measure ρ_σ is finite due to the construction $\triangleright$ ∎

Definition 2.4. A kernel $\hat{\mathrm{K}} \in \mathcal{H}_- \times \mathcal{H}_-$ satisfying the conditions (2.5) will be called **self-similar congruent** with respect to a given operator $L \in \mathcal{L}(\mathcal{H})$.

The construction done above for a self-similar congruent kernels $\hat{\mathrm{K}} \in \mathcal{H}_- \otimes \mathcal{H}_-$ in the form (2.7) subject to a given operator $\mathcal{L}(\mathcal{H})$ appears to be very inspiring if the condition self-similarity to make changed by a simple similarity. This topic will be discussed below.

3. Congruent kernel operators, related Delsarte transmutation mappings and their structure

3.1. Consider in a Hilbert space $\mathcal{H}$ a pair of densely defined linear differential operators L and $\tilde{L} \in \mathcal{L}(\mathcal{H})$. The following definition will be useful.

Definition 3.1. Let a pair of kernels $\hat{\mathrm{K}}_{\mathrm{s}} \in \mathcal{H}_- \otimes \mathcal{H}_-$, $s = \pm$, satisfy the following congruence relationships

$$(\tilde{L}_{ext} \otimes 1)\hat{\mathrm{K}}_{\mathrm{s}} = (1 \otimes L^*_{ext})\hat{\mathrm{K}}_{\mathrm{s}} \tag{3.1}$$

for a given pair of the correspondingly extended linear operators $L, \ \tilde{L} \in \mathcal{L}(\mathcal{H})$. Then the kernels $\hat{\mathrm{K}}_{\mathrm{s}} \in \mathcal{H}_- \otimes \mathcal{H}_-$, $s = \pm$, will be called congruent to this pair $(L, \tilde{L})$ of operators in $\mathcal{H}$.

Since not any pair of operators $L, \ \tilde{L} \in \mathcal{L}(\mathcal{H})$ can be congruent, the natural problem arises if they exist: how to describe the set of corresponding kernels $\hat{\mathrm{K}}_{\mathrm{s}} \in \mathcal{H}_- \otimes \mathcal{H}_-$, $s = \pm$, congruent to a given pair $(L, \tilde{L})$ of operators in $\mathcal{H}$. The first question being important for further is that of existence of kernels $\hat{\mathrm{K}}_{\mathrm{s}} \in \mathcal{H}_- \otimes \mathcal{H}_-$, $s = \pm$, congruent to this pair. The question has an evident answer for the case when $\tilde{L} = L$ and the congruence is then self- similar. The interesting case when $\tilde{L} \neq L$ appears to be very nontrivial and can be treated more or less successfully if there exist such bounded and invertible operators $\Omega_{\mathrm{s}} \in \mathcal{H}$, $s = \pm$, that the transmutation conditions

$$\tilde{L}\mathbf{\Omega}_{\mathrm{s}} = \mathbf{\Omega}_{\mathrm{s}} L \tag{3.2}$$

hold.

Definition 3.2. (Delsarte, Lions; [1, 2]) Let a pair of densely defined differential closeable operators $L, \tilde{L} \in \mathcal{L}(\mathcal{H})$ in a Hilbert space $\mathcal{H}$ is endowed with a pair of closed subspaces $\mathcal{H}_0, \tilde{\mathcal{H}}_0 \subset \mathcal{H}_-$ subject to a rigged Hilbert spaces chain (1.1). Then invertible operators $\Omega_{\mathrm{s}} \in Aut(\mathcal{H}) \cap \mathcal{B}(\mathcal{H})$, $s = \pm$, are called **Delsarte transmutations** if the following conditions hold:

- i) the operator $\mathbf{\Omega}_{\mathrm{s}}$ and its inverse $\mathbf{\Omega}_{\mathrm{s}}^{-1}$, $s = \pm$, are continuous in $\mathcal{H}$, that is $\Omega_{\mathrm{s}} \in Aut(\mathcal{H}) \cap \mathcal{B}(\mathcal{H}), s = \pm$;
- ii) the images $\mathrm{Im}\ \Omega_{\mathrm{s}}|_{\mathcal{H}_0} = \tilde{\mathcal{H}}_0, s = \pm$;
- iii) the relationships (3.2) are satisfied.

Suppose now that an operator pair $(L, \tilde{L}) \subset \mathcal{L}(\mathcal{H})$ is differential of the same order $n(L) \in \mathbb{Z}_+$, that is the following representations

$$L := \sum_{|\alpha|=0}^{n(L)} a_\alpha(x) \frac{\partial^{|\alpha|}}{\partial x^\alpha}, \qquad \tilde{L} := \sum_{|\alpha|=0}^{n(L)} \tilde{a}_\alpha(x) \frac{\partial^{|\alpha|}}{\partial x^\alpha}, \tag{3.3}$$

hold, where $x \in \mathrm{Q}$, $\mathrm{Q} \subset \mathbb{R}^m$ is some open connected region in $\mathbb{R}^m$, the coefficients a_α, $\tilde{a}_\alpha \in \mathcal{S}(\mathrm{Q}; End\ \mathbb{C}^{\mathrm{N}})$ for all $\alpha \in \mathbb{Z}_+^m$, $|\alpha| = \overline{0, n(L)}$ and $\mathrm{N} \in \mathbb{Z}_+$. The differential expressions (3.3) are defined and closeable on the dense in the Hilbert space $\mathcal{H} := L_2(\mathrm{Q}; \mathbb{C}^{\mathrm{N}})$ domains $\mathrm{D}(L)$, $\mathrm{D}(\tilde{L}) \subset \mathrm{W}_2^{n(L)}(\mathrm{Q}; \mathbb{C}^{\mathrm{N}}) \subset \mathcal{H}$. This, in particular, means that there exists the corresponding to (3.3) pair of adjoint operators $L^*, \tilde{L}^* \in \mathcal{L}(\mathcal{H})$ which are defined also on dense domains $\mathrm{D}(L^*)$, $\mathrm{D}(\tilde{L}^*) \subset \mathrm{W}_2^{n(L)}(\mathrm{Q}; \mathbb{C}^{\mathrm{N}}) \subset \mathcal{H}$.

Take now a pair of invertible bounded operators $\boldsymbol{\Omega}_{\mathrm{s}} \in Aut(\mathcal{H}) \cap \mathcal{B}(\mathcal{H})$, $s = \pm$, and look at the following Delsarte transformed operators

$$\tilde{L}_{\mathrm{s}} := \boldsymbol{\Omega}_{\mathrm{s}} L \boldsymbol{\Omega}_{\mathrm{s}}^{-1}, \tag{3.4}$$

$s = \pm$, which, by definition, must persist to be also differential. An additional natural constraint involved on operators $\Omega_s \in Aut(\mathcal{H}) \cap \mathcal{B}(\mathcal{H})$, $s = \pm$, is the independence [5, 12] of differential expressions for operators (3.4) on indices $s = \pm$. The problem of constructing such Delsarte transmutation operators $\Omega_s \in Aut(\mathcal{H}) \cap \mathcal{B}(\mathcal{H})$, $s = \pm$, appeared to be very complicated and in the same time dramatic as it one could observe from special results obtained in [5, 17] for two-dimensional Dirac and three-dimensional Laplace type operators.

3.2. Before proceeding to setting up our approach to treating the problem mentioned above, let us consider some formal generalizations of the results described before in Chapter 2. Take an elementary kernel $\hat{\tilde{Z}}_\lambda \in \mathcal{H}_- \otimes \mathcal{H}_-$ satisfying the conditions generalizing (2.4):

$$(\tilde{L}_{ext} \otimes \mathrm{I})\ \hat{\tilde{\mathrm{Z}}}_\lambda = \lambda \hat{\tilde{\mathrm{Z}}}_\lambda, \qquad (\mathrm{I} \otimes L_{ext}^*)\ \hat{\tilde{\mathrm{Z}}}_\lambda = \lambda \hat{\tilde{\mathrm{Z}}}_\lambda \tag{3.5}$$

for $\lambda \in \sigma(\tilde{L}) \cap \bar{\sigma}(L^*)$, being, evidently, well suitable for treating the equation (3.1). Then one sees that an elementary kernel $\widehat{\mathrm{Z}}_\lambda \in \mathcal{H}_- \otimes \mathcal{H}_-$ for any $\lambda \in \sigma(\widetilde{L}) \cap \sigma(\widetilde{L}^*)$ solves the equation (3.1), that is

$$(\widetilde{L}_{ext} \otimes 1)\ \hat{\tilde{\mathrm{Z}}}_\lambda = (1 \otimes L_{ext}^*)\ \hat{\tilde{\mathrm{Z}}}_\lambda. \tag{3.6}$$

Thereby one can expect that for kernels $\hat{\mathrm{K}}_{\mathrm{s}} \in \mathcal{H}_- \otimes \mathcal{H}_-$, $s = \pm$, there exist the similar to (2.9) spectral representations

$$\hat{\mathrm{K}}_{\mathrm{s}} = \int_{\sigma(\widetilde{L}) \cap \bar{\sigma}(L^*)} \hat{\tilde{\mathrm{Z}}}_\lambda \mathrm{d}\rho_{\sigma,\mathrm{s}}(\lambda), \tag{3.7}$$

$s = \pm$, with finite spectral measures $\rho_{\sigma,\mathrm{s}}$, $s = \pm$, localized upon the Borel subsets of the common spectrum $\sigma(\tilde{L}) \cap \bar{\sigma}(L^*)$. Based of the spectral representation like (2.8) applied separately to operators $\tilde{L} \in \mathcal{L}(\mathcal{H})$ and $L^* \in \mathcal{L}(\mathcal{H})$ one states similarly as before the following theorem.

Theorem 3.3. *The equations (3.5) are compatible for any $\lambda \in \sigma(\widetilde{L}) \cap \bar{\sigma}(L^*)$ and, moreover, for kernels $\hat{\mathrm{K}}_{\mathrm{s}} \in \mathcal{H}_- \otimes \mathcal{H}_-$, $s = \pm$, satisfying the congruence condition (3.1) there exist a kernel $\hat{\tilde{\mathrm{Z}}}_\lambda \in \mathcal{H}_{-\,-} \otimes \mathcal{H}_{-\,-}$ for a suitably Gelfand rigged Hilbert spaces chain (2.11), such that the spectral representations (3.7) hold.*

Now we will be interested in the inverse problem of constructing kernels $\hat{\mathrm{K}}_s \in \mathcal{H}_- \otimes \mathcal{H}_-$, $s = \pm$, like (3.7) a priori satisfying the congruence conditions (3.1) subject to the same pair $(L, \tilde{L})$ of differential operators in $\mathcal{H}$ and related via the Delsarte transmutation condition (3.2). In some sense we shall state that only for such Delsarte related operator pairs $(L, \tilde{L})$ in $\mathcal{H}$ one can construct a dual pair $\{\hat{\mathrm{K}}_s \in$

$\mathcal{H}_- \otimes \mathcal{H}_- : s = \pm\}$ of the corresponding congruent kernels satisfying the conditions like (3.1), that is

$$(\tilde{L}_{ext} \otimes 1)\ \hat{\mathrm{K}}_{\pm} = \hat{\mathrm{K}}_{\pm}(1 \otimes L^*_{ext}). \tag{3.8}$$

3.3. Suppose now that there exists another pair of Delsarte transmutation operators $\mathbf{\Omega}_s$ and $\mathbf{\Omega}^{\circledast}_s \in Aut(H) \cap \mathcal{B}(\mathcal{H})$, $s = \pm$, satisfying condition ii) of Definition 3.2 subject to the corresponding two pairs of differential operators $(L, \tilde{L})$ and $(L^*, \tilde{L}^*) \subset \mathcal{L}(\mathcal{H})$. This means, in particular, that there exists an additional pair of closed subspace $\mathcal{H}^{\circledast}_0$ and $\tilde{\mathcal{H}}^{\circledast}_0 \subset \mathcal{H}_-$ such that

$$\mathrm{Im}\ \mathbf{\Omega}^{\circledast}_s|_{\mathcal{H}^{\circledast}_0} = \tilde{\mathcal{H}}^{\circledast}_0 \tag{3.9}$$

$s = \pm$, for the Delsarte transmutation operator $\mathbf{\Omega}^{\circledast}_s \in Aut(H) \cap \mathcal{B}(\mathcal{H})$, $s = \pm$, satisfying the obvious conditions

$$\tilde{L}^* \cdot \mathbf{\Omega}^{\circledast}_s = \mathbf{\Omega}^{\circledast}_s \cdot L^* \tag{3.10}$$

$s = \pm$, involving the adjoint operators $\tilde{L}^*, L^* \in \mathcal{L}(\mathcal{H})$ defined before and given by the following from (3.3) usual differential expressions:

$$L^* = \sum_{|\alpha|=0}^{n(L)} (-1)^{|\alpha|} \frac{\partial^{|\alpha|}}{\partial x^\alpha} \bar{a}^{\intercal}_\alpha(x),\ \ \tilde{L}^* = \sum_{|\alpha|=0}^{n(L)} (-1)^{|\alpha|} \frac{\partial^{|\alpha|}}{\partial x^\alpha} \bar{\tilde{a}}^{\intercal}_\alpha(x) \tag{3.11}$$

for all $x \in \mathrm{Q} \subset \mathbb{R}^m$.

Construct now the following [18, 19] Delsarte transmutation operators of Volterra type

$$\mathbf{\Omega}_{\pm} := 1\ + \mathrm{K}_{\pm}(\mathbf{\Omega}), \tag{3.12}$$

corresponding to some two different kernels $\hat{\mathrm{K}}_+$ and $\hat{\mathrm{K}}_- \in \mathcal{H}_- \otimes \mathcal{H}_-$, of integral Volterrian operators $\mathrm{K}_+(\mathbf{\Omega})$ and $\mathrm{K}_-(\mathbf{\Omega})$ related with them in the following way:

$$(u, \mathrm{K}_{\pm}(\mathbf{\Omega})v) := (u\ \chi(S^{(m)}_{x,\pm}), \hat{\mathrm{K}}_{\pm}v) \tag{3.13}$$

for all $(u, v) \in \mathcal{H}_+ \times \mathcal{H}_+$, where $\chi(S^{(m)}_{x,\pm})$ are some characteristic functions of two m-dimensional smooth hypersurfaces $S^{(m)}_{x,+}$ and $S^{(m)}_{x,-} \in \mathcal{K}(\mathrm{Q})$ from a singular simplicial complex $\mathcal{K}(\mathrm{Q})$ of the open set $\mathrm{Q} \subset \mathbb{R}^m$, chosen such that the boundary $\partial(S^{(m)}_{x,+} \cup S^{(m)}_{x,-}) = \partial\mathrm{Q}$. In the case when $Q := \mathbb{R}^m$, it is assumed naturally that $\partial\mathbb{R}^m = \oslash$. Making use of the Delsarte operators (3.12) and relationship like (3.2) one can construct the following differential operator expressions:

$$\tilde{L}_{\pm} - L = \mathrm{K}_{\pm}(\mathbf{\Omega})L - \tilde{L}_{\pm}\mathrm{K}_{\pm}(\mathbf{\Omega}). \tag{3.14}$$

Since the left-hand sides of (3.14) are, by definition, purely differential expressions, one follows right away that the local kernel relationships like (3.7) hold:

$$(\tilde{L}_{ext,\pm} \otimes 1)\ \hat{\mathrm{K}}_{\pm} = (1 \otimes L^*_{ext})\ \hat{\mathrm{K}}_{\pm}. \tag{3.15}$$

The expressions (3.14) define, in general, two different differential expressions $\tilde{L}_{\pm} \in \mathcal{L}(\mathcal{H})$ depending correspondingly both on the kernels $\hat{\mathrm{K}}_{\pm} \in \mathcal{H}_- \otimes \mathcal{H}_-$ and on the chosen hypersurfaces $S^{(m)}_{x,\pm} \in \mathcal{K}(\mathrm{Q})$. As will be stated later, the following important theorem holds.

Theorem 3.4. *Let smooth hypersurfaces* $S_{x,\pm}^{(m)} \in \mathcal{K}(\mathrm{Q})$ *be chosen in such a way that* $\partial(S_{x,+}^{(m)} \cup S_{x,-}^{(m)}) = \partial \mathrm{Q}$ *and* $\partial S_{x,\pm}^{(m)} = \mp\sigma_x^{(m-1)} + \sigma_{x\pm}^{(m-1)}$, *where* $\sigma_x^{(m-1)}$ *and* $\sigma_{x\pm}^{(m-1)}$ *are some homological subject to the the homology group* $\mathrm{H}_{m-1}(\mathrm{Q};\mathbb{C})$ *simplicial chains, parametrized, correspondingly, by a running point* $x \in \mathrm{Q}$ *and fixed points* $x_\pm \in \partial\mathrm{Q}$ *and satisfying the following homotopy condition:* $\lim_{x\to x_\pm} \ \sigma_x^{(m-1)} = \mp\sigma_{x_\pm}$. *Then the operator equalities*

$$\tilde{L}_+ := \mathbf{\Omega}_+ L \mathbf{\Omega}_+^{-1} = \tilde{L} = \mathbf{\Omega}_- L \mathbf{\Omega}_-^{-1} := \tilde{L}_- \tag{3.16}$$

are satisfied if the following commutation property

$$[\mathbf{\Omega}_+^{-1} \ \mathbf{\Omega}_-, L] = 0 \tag{3.17}$$

or, equivalently, kernel relationship

$$(L_{ext} \otimes 1)\hat{\mathbf{\Omega}}_+^{-1} * \hat{\mathbf{\Omega}}_- = (1 \otimes L_{ext}^*)\hat{\mathbf{\Omega}}_+^{-1} * \hat{\mathbf{\Omega}}_- \tag{3.18}$$

hold.

Remark 3.5. It is a place to notice here that special degenerate cases of theorem 3.4 where before proved in works [5, 17] for two-dimensional Dirac and three-dimensional Laplace type differential operators. The constructions and tools devised there appeared to be instructive and motivative for the approach developed here by us in the general case.

3.4. Consider now a pair $(\mathbf{\Omega}_+, \mathbf{\Omega}_-)$ of Delsarte transmutation operators being in the form (3.12) and respecting all of the conditions from Theorem 3.4. Then the following lemma is true.

Lemma 3.6. *Let an invertible Fredholm operator* $\mathbf{\Omega} = 1 + \Phi(\mathbf{\Omega}) \in Aut(\mathcal{H}) \cap \mathcal{B}(\mathcal{H})$ *with* $\Phi \in \mathcal{B}_\infty(\mathcal{H})$ *allow the factorization representation*

$$\mathbf{\Omega} = \mathbf{\Omega}_+^{-1} \mathbf{\Omega}_- \tag{3.19}$$

by means of two Delsarte operators $\mathbf{\Omega}_+$ *and* $\mathbf{\Omega}_- \in Aut(\mathcal{H}) \cap \mathcal{B}(\mathcal{H})$ *in the form (3.12). Then there exists the unique operator kernel* $\hat{\Phi} \in \mathcal{H}_- \otimes \mathcal{H}_-$ *corresponding naturally to the compact operator* $\Phi(\mathbf{\Omega}) \in \mathcal{B}_\infty(\mathcal{H})$ *and satisfying the following self-similar congruence commutation condition:*

$$(L_{ext} \otimes 1) \ \hat{\Phi} = (1 \otimes L_{ext}^*) \ \hat{\Phi}, \tag{3.20}$$

related to the properties (3.17 and (3.18).

From the equality (3.20) and Theorem 2.2 one gets easily the following corollary.

Corollary 3.7. There exists such a finite Borel measure ρ_σ defined on the Borel subsets of $\sigma(\mathrm{L}) \cap \overline{\sigma}(\mathrm{L}^*)$, that the following weak equality

$$\widehat{\Phi} = \int_{\sigma(\mathrm{L}) \cap \overline{\sigma}(\mathrm{L}^*)} \widehat{\mathrm{Z}}_\lambda \mathrm{d}\rho_\sigma(\lambda) \tag{3.21}$$

holds.

Concerning the differential expression $L \in \mathcal{L}(\mathcal{H})$ and the corresponding Volterra type Delsarte transmutation operators $\mathbf{\Omega}_\pm \in \mathcal{B}_\infty(\mathcal{H})$ the conditions (3.17) and (3.20) are equivalent to the operator equation

$$[\Phi(\mathbf{\Omega}), L] = 0. \tag{3.22}$$

Really, since equalities (3.16) hold, one gets easily that

$$L(1+\Phi(\mathbf{\Omega}))=L(\mathbf{\Omega}_+^{-1}\mathbf{\Omega}_-)=\mathbf{\Omega}_+^{-1}(\mathbf{\Omega}_+L\mathbf{\Omega}_+^{-1})\mathbf{\Omega}_-$$

$$=\mathbf{\Omega}_+^{-1}(\mathbf{\Omega}_-L\mathbf{\Omega}_-)\mathbf{\Omega}_-=\mathbf{\Omega}_+^{-1}\mathbf{\Omega}_-L=(1+\Phi(\mathbf{\Omega}))L, \tag{3.23}$$

meaning exactly (3.22).

Suppose also that, first, for another Fredholm operator $\mathbf{\Omega}^{\circledast}=1+\Phi^{\circledast}(\mathbf{\Omega})\in Aut(\mathcal{H})\cap\mathcal{B}(\mathcal{H})$ with $\Phi^{\circledast}(\mathbf{\Omega})\in\mathcal{B}_\infty(\mathcal{H})$ there exist two factorizing it Delsarte transmutation Volterra type operators $\mathbf{\Omega}_\pm^{\circledast}\in Aut(\mathcal{H})\cap\mathcal{B}(\mathcal{H})$ in the form

$$\mathbf{\Omega}_\pm^{\circledast}=1+\mathrm{K}_\pm^{\circledast}(\mathbf{\Omega}) \tag{3.24}$$

with Volterrian [18] integral operators $\mathrm{K}_\pm^{\circledast}(\mathbf{\Omega})$ related naturally with some kernels $\hat{\mathrm{K}}_\pm\in\mathcal{H}_-\otimes\mathcal{H}_-$, and, second, the factorization condition

$$1+\Phi^{\circledast}(\mathbf{\Omega})=\mathbf{\Omega}_+^{\circledast,-1}\mathbf{\Omega}_-^{\circledast} \tag{3.25}$$

is satisfied, then the following theorem holds.

Theorem 3.8. *Let a pair of hypersurfaces $S_{x,\pm}^{(m)}\subset\mathcal{K}(\mathrm{Q})$ satisfy all of the conditions from Theorem 3.2. Then the Delsarte transformed operators $\tilde{L}_\pm^*\in\mathcal{L}(\mathcal{H})$ are differential and equal, that is*

$$\tilde{L}_+^*=\mathbf{\Omega}_+^{\circledast}L^*\mathbf{\Omega}_+^{\circledast,-1}=\tilde{L}^*=\mathbf{\Omega}_-^{\circledast}L^*\mathbf{\Omega}_-^{\circledast,-1}=\tilde{L}_-^{\circledast}, \tag{3.26}$$

iff the following commutation condition

$$[\Phi^{\circledast}(\mathbf{\Omega}),L^*]=0 \tag{3.27}$$

holds.

Proof. $\triangleleft$ A proof of this theorem is stated by reasonings similar to those done before when analyzing the congruence condition for a given pair $(L,\tilde{L})\subset\mathcal{L}(\mathcal{H})$ of differential operators and their adjoint ones in $\mathcal{H}$. $\triangleright$ ∎

By means of the Delsarte transmutation from the differential operators L and $L^*\in\mathcal{L}(\mathcal{H})$ we have obtained above two differential operators

$$\tilde{L}=\mathbf{\Omega}_\pm L\mathbf{\Omega}_\pm^{-1},\quad \tilde{L}^*=\mathbf{\Omega}_\pm^{\circledast}L^*\mathbf{\Omega}_\pm^{\circledast,-1}, \tag{3.28}$$

which must be compatible and, thereby, related as

$$(\tilde{L})^*=(\widetilde{L^*}). \tag{3.29}$$

The condition (3.29) due to (3.28) gives rise to the following additional commutation expressions for kernels $\mathbf{\Omega}_\pm^{\circledast}$ and $\mathbf{\Omega}_\pm^*\in Aut(\mathcal{H})\cap\mathcal{B}(\mathcal{H})$:

$$[L^*,\mathbf{\Omega}_\pm^*\mathbf{\Omega}_\pm^{\circledast}]=0, \tag{3.30}$$

being equivalent, obviously, to such a commutation relationship:

$$[L,\mathbf{\Omega}_\pm^{\circledast,*}\mathbf{\Omega}_\pm]=0. \tag{3.31}$$

As a result of representations (3.31) one can formulate the following corollary.

Corollary 3.9. There exist finite Borel measures $\rho_{\sigma,\pm}$ localized upon the common spectrum $\sigma(L)\cap\bar{\sigma}(L^*)$, such that the following weak kernel representations

$$\hat{\mathbf{\Omega}}_{\pm}^{\circledast,*}*\hat{\mathbf{\Omega}}_{\pm}=\int_{\sigma(L)\cap\bar{\sigma}(L^*)}\hat{\mathrm{Z}}_{\lambda}\mathrm{d}\rho_{\sigma,\pm}(\lambda) \tag{3.32}$$

hold, where $\hat{\mathbf{\Omega}}_{\pm}^{\circledast,*}$ and $\hat{\mathbf{\Omega}}_{\pm}\in\mathcal{H}_{-}\otimes\mathcal{H}_{-}$ are the corresponding kernels of integral Volterrian operators $\mathbf{\Omega}_{\pm}^{\circledast,*}$ and $\mathbf{\Omega}_{\pm}\in Aut(\mathcal{H})\cap\mathcal{B}(\mathcal{H})$.

3.5. The integral operators of Volterra type (3.12) constructed above by means of kernels in the form (3.7) are, as well known [13, 7, 5, 11, 10], very important for studying many problems of spectral analysis and related integrable nonlinear dynamical systems [12, 11, 6, 17, 25] on functional manifolds. In particular, they serve as factorizing operators for a class of Fredholm operators entering the fundamental Gelfand - Levitan - Marchenko operator equations [13, 11, 12] whose solutions are exactly kernels of Delsarte transmutation operators of Volterra type, related with the corresponding congruent kernels subject to given pairs of closeable differential operators in a Hilbert space $\mathcal{H}$. Thereby it is natural to try to learn more of their structure properties subject to their representations both in the form (3.7), (3.12), and in the dual form within the general Gokhberg - Krein theory [18, 12, 19] of Volterra type operators.

To proceed further with we need to introduce some additional notions and definitions from [18, 10] important for what will follow below. Define a set $\mathcal{P}$ of projectors $\mathrm{P}^2=\mathrm{P}:\mathcal{H}\ \to\mathcal{H}$ which is called a **projector chain** if for any pair $\mathrm{P}_1,\mathrm{P}_2\in\mathcal{P}$, $\mathrm{P}_1\neq\mathrm{P}_2$, one has either $\mathrm{P}_1<\mathrm{P}_2$ or $\mathrm{P}_2<\mathrm{P}_1$, and $\mathrm{P}_1\mathrm{P}_2=\min(\mathrm{P}_1,\mathrm{P}_2)$. The ordering $\mathrm{P}_1<\mathrm{P}_2$ above means, as usually, that $\mathrm{P}_1\mathcal{H}\subset\mathrm{P}_2\mathcal{H}$, $\mathrm{P}_1\mathcal{H}\neq\mathrm{P}_2\mathcal{H}$. If $\mathrm{P}_1\mathcal{H}\subset\mathrm{P}_2\mathcal{H}$, then one writes down that $\mathrm{P}_1\leq\mathrm{P}_2$. The closure $\overline{\mathcal{P}}$ of a chain $\mathcal{P}$ means, by definition, that set of all operators being weak limits of sequences from $\mathcal{P}$. The inclusion relationship $\mathcal{P}_1\subset\mathcal{P}_2$ of any two sets of projector chains possesses obviously the transitivity property allowing to consider the set of all projector chains as a partly ordered set. A chain $\mathcal{P}$ is called **maximal** if it can not be extended. It is evident that a maximal chain is closed and contains zero $0\in\mathcal{P}$ and unity $1\in\mathcal{P}$ operators. A pair of projectors $(\mathrm{P}^{-},\mathrm{P}^{+})\subset\mathcal{P}$ is called a **break** of the chain $\mathcal{P}$ if $\mathrm{P}_{-}<\mathrm{P}_{+}$ and for all $\mathrm{P}\in\mathcal{P}$ either $\mathrm{P}<\mathrm{P}^{-}$ or $\mathrm{P}^{+}<\mathrm{P}$. A closed chain is called **continuous** if for any pair of projectors $\mathrm{P}_1,\mathrm{P}_2\subset\mathcal{P}$ there exist a projector $\mathrm{P}\in\mathcal{P}$, such that $\mathrm{P}_1<\mathrm{P}<\mathrm{P}_2$. A maximal chain $\mathcal{P}$ will be called complete if it is continuous. A strongly ascending with to inclusion projector valued function $\mathrm{P}:\mathrm{Q}\ni\Delta\ \to\ \mathcal{P}$ is called a **parametrization** of a chain $\mathcal{P}$, if the chain $\mathcal{P}=Im(\mathbb{P})$ such a parametrization of the self-adjoint chain $\mathcal{P}$ is called **smooth**, if for any $u\in\mathcal{H}$ the positive value measure $\Delta\ \to\ (u,\mathrm{P}(\Delta)u)$ is absolutely continuous. It is well known [18, 19, 12] that very complete projector chain allows a smooth parametrization. In what will follow a projector chain $\mathcal{P}$ will be self-adjoint, complete and endowed with a fixed smooth parametrization with respect to an operator valued function $\mathrm{F}:\mathcal{P}\ \to\ \mathcal{B}(\mathcal{H})$ the expressions like $\int_{\mathcal{P}}\mathrm{F}(\mathrm{P})\mathrm{dP}$ and $\int_{\mathcal{P}}\mathrm{dPF}(\mathrm{P})$ will be used for the corresponding [18] Riemann-Stiltjes integrals subject to the corresponding projector chain. Take now a linear compact operator $\mathrm{K}\in\mathcal{B}_{\infty}(\mathcal{H})$ acting in a separable Hilvert space $\mathcal{H}$ endowed with a projector chain $\mathcal{P}$. A chain $\mathcal{P}$ is also called **proper** subject to an operator $\mathrm{K}\in\mathcal{B}_{\infty}(\mathcal{H})$ if $\mathrm{PKP}=\mathrm{KP}$ for any projector $\mathrm{P}\in\mathcal{P}$, meaning obviously that subspace $\mathrm{P}\mathcal{H}$ is invariant with respect

to the operator $\mathrm{K} = \mathcal{B}_\infty(\mathcal{H})$ for any $\mathrm{P} \in \mathcal{P}$. As before the note by $\sigma(\mathrm{K}(\mathbf{\Omega}))$. The spectrum of any operator $\mathrm{K} \in \mathcal{L}(\mathcal{H})$.

Definition 3.10. An operator $\mathrm{K} \in \mathcal{B}_\infty(\mathcal{H})$ is called **Volterrian** if $\sigma(\mathrm{K}) = \{0\}$.

As it can be shown [18], a Volterrian operator $\mathrm{K} \in \mathcal{B}_\infty(\mathcal{H})$ possesses the maximal proper projector chain $\mathcal{P}$ such, that for any its break $(\mathrm{P}^-, \mathrm{P}^-)$ the following relationship

$$(\mathrm{P}^+ - \mathrm{P}^-)\ \mathrm{K}\ (\mathrm{P}^+ - \mathrm{P}^-) = 0 \tag{3.33}$$

holds. Since integral operators (3.12) constructed before are of Volterra type and congruent to a pair $(L, \widetilde{L})$ of closeable differential operators in $\mathcal{H}$, we will be now interested in their properties with respect both to the definition given above and to the corresponding proper maximal projector chains $\mathcal{P}(\mathbf{\Omega})$.

3.6. Suppose now that we are given a Fredholm operator $\mathbf{\Omega} \in \mathcal{B}(\mathcal{H}) \cap Aut(\mathcal{H})$ self- congruent to a closeable differential operator $L \in \mathcal{L}(\mathcal{H})$. As we are also given with an elementary kernel (2.6) in the spectral form (2.7), our present task will be a description of elementary kernels $\hat{\mathrm{Z}}_\lambda,\ \lambda \in \sigma(L) \cap \bar{\sigma}(L^*)$, by means of some smooth and complete parametrization suitable for them. For treating this problem we will make use of very interesting recent results obtained in [19] and devoted to the factorization problem of Fredholm operators. As a partial case this work contains some aspects of our factorization problem for Delsarte transmutation operators $\mathbf{\Omega} \in Aut(\mathcal{H}) \cap \mathcal{B}(\mathcal{H})$ in the form (3.12).

Let us formulate now some preliminary results from [18, 19] suitable for the problem under regard. As before, we will the note by $\mathcal{B}(\mathcal{H})$ the Banach algebra of all linear and continuous every where defined operators in $\mathcal{H}$, and also by $\mathcal{B}_\infty(\mathcal{H})$ the Banach algebra of all compact operators from $\mathcal{B}(\mathcal{H})$ and by $\mathcal{B}_0(\mathcal{H})$ the linear subspace of all finite dimensional operator from $\mathcal{B}_\infty(\mathcal{H})$.
Put also, by definition,

$$\begin{aligned} \mathcal{B}^-(\mathcal{H}) &= \{\mathrm{K} \in \mathcal{B}(\mathcal{H}) : (1-\mathrm{P})\mathrm{K}\mathrm{P} = 0,\ \mathrm{P} \in \mathcal{P}\}, \\ \mathcal{B}^+(\mathcal{H}) &= \{\mathrm{K} \in \mathcal{B}(\mathcal{H}) : \mathrm{P}\mathrm{K}(1-\mathrm{P}) = 0,\ \mathrm{P} \in \mathcal{P}\} \end{aligned} \tag{3.34}$$

and call an operator $\mathrm{K} \in \mathcal{B}^+$, $(\mathrm{K} \in \mathcal{B}^-)$ **up-triangle** (**down-triangle**) with respect to the projector chain $\mathcal{P}$. Denote also by $\mathcal{B}_p(\mathcal{H})$, $p \in [1, \infty]$, the so called Neumann-Shattin ideals and put

$$\mathcal{B}^+_\infty(\mathcal{H}) := \mathcal{B}_\infty(\mathcal{H}) \cap \mathcal{B}^+(\mathcal{H}),\ \ \mathcal{B}^-_\infty(\mathcal{H}) := \mathcal{B}_\infty(\mathcal{H}) \cap \mathcal{B}^-(\mathcal{H}). \tag{3.35}$$

Subject to Definition 3.4 Banach subspaces (3.35) are Volterrian,, being closed in $\mathcal{B}_\infty(\mathcal{H})$ and satisfying the condition

$$\mathcal{B}^+_\infty(\mathcal{H}) \cap \mathcal{B}^-_\infty(\mathcal{H}) = \varnothing. \tag{3.36}$$

Denote also by $\mathcal{P}^+$ $(\mathcal{P}^-)$ the corresponding projectors of the linear space

$$\widetilde{\mathcal{B}}_\infty(\mathcal{H}) := \mathcal{B}^+_\infty(\mathcal{H}) \oplus \mathcal{B}^-_\infty(\mathcal{H}) \subset \mathcal{B}_\infty(\mathcal{H})$$

upon $\mathcal{B}^+_\infty(\mathcal{H})(\ \mathcal{B}^-_\infty(\mathcal{H}))$, and call them after [18] by **transformators** of a **triangle shear**. The transformators $\mathcal{P}^+$ and $\mathcal{P}^-$ are known [18] to be continuous operators in ideals $\mathcal{B}_p(\mathcal{H})$, $p \in [1, \infty]$. From definitions above one gets that

$$\mathcal{P}^+(\Phi) + \mathcal{P}^-(\Phi) = \Phi,\ \ \mathcal{P}^\pm(\Phi) = \tau\mathcal{P}^\mp\tau(\Phi) \tag{3.37}$$

for any $\Phi \in \mathcal{B}(\mathcal{H})$, where $\tau : \mathcal{B}_p(\mathcal{H}) \ \to\ \mathcal{B}_p(\mathcal{H})$ is the standard involution in $\mathcal{B}_p(\mathcal{H})$ acting as $\tau(\Phi) := \Phi^*$.

Remark 3.11. It is clear and important that transformators $\mathcal{P}^{+}$ and $\mathcal{P}^{-}$ strongly depend on a fixed projector chain $\mathcal{P}$.

Put now, by definition,

$$\mathcal{V}_f^{\pm} := \{1+\mathrm{K}_{\pm} : \mathrm{K}_{\pm} \in \mathcal{B}_{\infty}^{\pm}(\mathcal{H})\} \tag{3.38}$$

and

$$\mathcal{V}_f := \{\boldsymbol{\Omega}_{+}^{-1}\cdot\boldsymbol{\Omega}_{-} : \boldsymbol{\Omega}_{\pm} \in \mathcal{V}_f^{\pm}\}. \tag{3.39}$$

It is easy to check that $\mathcal{V}_f^{+}$ and $\mathcal{V}_f^{-}$ are subgroups of invertible operators from $Aut(\mathcal{H})\cap\mathcal{B}(\mathcal{H})$ and, moreover, $\mathcal{V}_f^{+}\cap\mathcal{V}_f^{-} = \{1\}$. Consider also the following two operator sets:

$$\mathcal{W} := \{\Phi \in \mathcal{B}_{\infty}(\mathcal{H}) : \ \mathrm{Ker}(1+\mathrm{P}\Phi\mathrm{P}) = \{0\}, \quad \mathrm{P}\in\mathcal{P}\},$$

$$\mathcal{W}_f := \{\Phi \in \mathcal{B}_{\infty}(\mathcal{H}) : \quad \boldsymbol{\Omega} := 1+\Phi \in \mathcal{V}_f\}, \tag{3.40}$$

which are characterized by the following (see [18, 19]) theorem.

Theorem 3.12 (I.C. Gokhberg and M.G. Krein)**.** *The following conditions hold:*

i) $\mathcal{W}_f \subset \mathcal{W}$;
ii) $\mathcal{B}_{\omega}(\mathcal{H})\cap\mathcal{W} \subset \mathcal{W}_f$ *where* $\mathcal{B}_{\omega}(\mathcal{H}) \subset \mathcal{B}(\mathcal{H})$ *is the so called Macaev ideal;*
iii) *for any* $\Phi\in\mathcal{W}_f$ *it is necessary and sufficient that at least one of integrals*

$$\mathcal{K}_{+}(\boldsymbol{\Omega}) = -\int_{\mathcal{P}} \mathrm{d}\mathrm{P}\Phi\mathrm{P}(1+\mathrm{P}\Phi\mathrm{P})^{-1},$$

$$(1+\mathrm{K}_{-}(\boldsymbol{\Omega}))^{-1} - 1 = -\int_{\mathcal{P}} (1+\mathrm{P}\Phi\mathrm{P})^{-1}\mathrm{P}\Phi\mathrm{d}\mathrm{P} \tag{3.41}$$

is convergent in the uniform operator topology, and, moreover, if the one integral of (3.24) is convergent then the another one is convergent too;

iiii) *the factorization representation*

$$\boldsymbol{\Omega} = 1+\Phi = (1+\mathrm{K}_{+}(\boldsymbol{\Omega}))^{-1}(1+\mathrm{K}_{-}(\boldsymbol{\Omega})) \tag{3.42}$$

for $\Phi\in\mathcal{W}_f$ *is satisfied.*

The theorem above is still abstract since it doesn't take into account the crucial relationship (3.22) relating the operators representation (3.42) with a given differential operator $L\in\mathcal{L}(\mathcal{H})$. Thus, it is necessary to satisfy the condition (3.22). If this condition is due to (3.1) and (3.16) satisfied, the following crucial equalities

$$(1+\mathrm{K}_{+}(\boldsymbol{\Omega}))L(1+\mathrm{K}_{+}(\boldsymbol{\Omega}))^{-1} = \tilde{L} = (1+\mathrm{K}_{-}(\boldsymbol{\Omega}))L(1+\mathrm{K}_{-}(\boldsymbol{\Omega}))^{-1} \tag{3.43}$$

in $\mathcal{H}$ and the corresponding congruence relationships

$$(\tilde{L}_{ext}\otimes 1)\widehat{\mathrm{K}}_{\pm} = (1\otimes L_{ext}^{*})\widehat{\mathrm{K}}_{\pm} \tag{3.44}$$

in $\mathcal{H}_{+}\otimes\mathcal{H}_{+}$ hold. Here by $\hat{\mathrm{K}}_{\pm}\in\mathcal{H}_{-}\otimes\mathcal{H}_{-}$ we denoted the corresponding kernels of Volterra operators $\mathrm{K}_{\pm}(\boldsymbol{\Omega}) \in \mathcal{B}_{\infty}^{\pm}(\mathcal{H})$.Since the factorization (3.42) is unique, the corresponding kernels must a priori satisfy the conditions (3.43) and (3.44). Thereby the self-similar congruence condition must be solved with respect to a kernel $\hat{\Phi}\in\mathcal{H}_{-}\otimes\mathcal{H}_{-}$ corresponding to the integral operator $\Phi\in\mathcal{B}_{\infty}(\mathcal{H})$, and next, must be found the corresponding unique factorization (3.42), satisfying a priori condition (3.43) and (3.44).

3.7 To realize this scheme define preliminarily a unique positive Borel finite measure on the Borel subsets $\Delta \subset \mathrm{Q}$ of the open set $\mathrm{Q} \subset \mathbb{R}^m$, satisfying for any projector $\mathrm{P}_x \in \mathcal{P}_x$ of a chain $\mathcal{P}_x$, marked by a running point $x \in \mathrm{Q}$, the following condition

$$(u, \mathrm{P}_x(\Delta)v)_{\mathcal{H}} = \int_{\Delta \subset Q} (u, \mathcal{X}_x(y)v)\mathrm{d}\mu_{\mathcal{P}_x}(y) \tag{3.45}$$

for all $u, v \in \mathcal{H}_+$, where $\mathcal{X}_x : \mathrm{Q} \to \mathcal{B}_2(\mathcal{H}_+, \mathcal{H}_-)$ is for any $x \in \mathrm{Q}$ a measurable with respect to some Borel measure $\mu_{\mathcal{P}_x}$ on Borel subsets of Q operator-valued mapping of Hilbert-Schmidt type. The representation (3.45) follows due the reasoning similar to that in [8], based on the standard Radon-Nikodym theorem [8, 32]. This means in particular, that in the weak sense

$$\mathrm{P}_x(\Delta) = \int_{\Delta} \mathcal{X}_x(y)\mathrm{d}\mu_{\mathcal{P}_x}(y) \tag{3.46}$$

for any Borel set $\Delta \in \mathrm{Q}$ and a running point $x \in \mathrm{Q}$. Making use now of the weak representation (3.46) the integral expression like $\mathrm{I}_{f,g}(x) = \int_{\mathcal{P}_x} f(\mathrm{P}_x)\mathrm{dP}_x g(\mathrm{P}_x)$, $x \in \mathrm{Q}$, for any continuous mappings $f, g : \mathcal{P}_x \to \mathcal{B}(\mathcal{H})$ can be, obviously, represented as

$$\mathrm{I}_{f,g}(x) = \int_{\mathrm{Q}} f(\mathrm{P}(y))\chi_x(y)g(\mathrm{P}(y))\mathrm{d}\mu_{\mathcal{P}_x}(y). \tag{3.47}$$

Thereby for the Volterrian operators (3.41) one can get the following expressions:

$$\mathrm{K}_{+,x}(\mathbf{\Omega}) = -\int_{\mathrm{Q}} (1 + \mathrm{P}_x(y)\Phi\mathrm{P}_x(y))^{-1}\mathrm{P}_x(y)\Phi\mathrm{d}\mu_{\mathcal{P}_{x,+}}(y),$$

$$(1 + \mathrm{K}_{+,x}(\mathbf{\Omega}))^{-1} = 1 - \int_{\mathrm{Q}} \mathrm{d}\mu_{\mathcal{P}_{x,+}}(y)\Phi\mathrm{P}_x(y)(1 + \mathrm{P}_x(y)\Phi\mathrm{P}_x(y))^{-1} \tag{3.48}$$

for some Borel measure $\mu_{\mathcal{P}_{x,+}}$ on Q and a given operator $\Phi \in \mathcal{B}_\infty(\mathcal{H})$. The first expression of (3.48) can be written down for the corresponding kernels $\hat{\mathrm{K}}_{+,x}(y) \in \mathcal{H}_- \otimes \mathcal{H}_-$ as follows

$$\hat{\mathrm{K}}_{+,x}(y) = -\int_{\sigma(L)\cap\overline{\sigma}(L^*)} \mathrm{d}\rho_{\sigma,+}(\lambda)\tilde{\psi}_\lambda(x) \otimes \varphi_\lambda(y), \tag{3.49}$$

where, due to the representation (3.21) and Theorem 2.2. we put for any running points x, y and $x' \in \mathrm{Q}$ the following convolution of two kernels:

$$((1 + \mathrm{P}_x(x')\Phi\mathrm{P}_x(x'))^{-1}) * (\psi_\lambda(x') \otimes \varphi_\lambda(y)) := \tilde{\psi}_\lambda(x) \otimes \varphi_\lambda(y). \tag{3.50}$$

for $\lambda \in \sigma(\tilde{L}) \cap \bar{\sigma}(L^*)$ and some $\tilde{\psi}_\lambda \in \mathcal{H}_-$. Taking now into account the representation (3.7) at $s = "+"$, from (3.49) one gets easily that the elementary congruent kernel

$$\hat{\hat{\mathrm{Z}}}_\lambda = \tilde{\psi}_\lambda \otimes \varphi_\lambda \tag{3.51}$$

satisfies the important conditions $(\tilde{L}_{ext} \otimes 1)\hat{\hat{\mathrm{Z}}}_\lambda = \lambda\hat{\hat{\mathrm{Z}}}_\lambda$ and $(1 \otimes L^*)\hat{\hat{\mathrm{Z}}}_\lambda = \lambda\hat{\hat{\mathrm{Z}}}_\lambda$ for any $\lambda \in \sigma(\tilde{L}) \cap \bar{\sigma}(L^*)$. Now for the operator $\mathrm{K}_+(\mathbf{\Omega}) \in \mathcal{B}^+_\infty(\mathcal{H})$ one finds the following integral representation

$$\mathrm{K}_+(\mathbf{\Omega}) = -\int_{S^{(m)}_{+,x}} \mathrm{d}y \int_{\sigma(\tilde{L})\cap\bar{\sigma}(L^*)} \mathrm{d}\rho_{\sigma,+}(\lambda)\tilde{\psi}_\lambda(x)\bar{\varphi}^{\intercal}_\lambda(y)(\cdot), \tag{3.52}$$

satisfying, evidently the congruence condition (3.1), where we put, by definition,

$$d\mu_{\mathcal{P}_{x,+}}(y) = \chi_{S^{(m)}_{+,x}} dy, \tag{3.53}$$

with $\chi_{S^{(m)}_{+,x}}$ being the characteristic function of the support of the measure $d\mu_{\mathcal{P}_x}$, that is $supp\ \ \mu_{\mathcal{P}_{x,+}} := S^{(m)}_{+,x} \in \mathcal{K}(\mathrm{Q})$. Completely similar reasonings can be applied for describing the structure of the second factorizing operator $\mathrm{K}_-(\mathbf{\Omega}) \in \mathcal{B}^-_\infty(\mathcal{H})$:

$$\mathrm{K}_-(\mathbf{\Omega}) = -\int_{S^{(m)}_{-,x}} dy \int_{\sigma(\tilde{L})\cap\bar{\sigma}(L^*)} d\rho_{\sigma,-}(\lambda)\tilde{\psi}_\lambda(x)\bar{\varphi}^\intercal_\lambda(y)(\cdot), \tag{3.54}$$

where, by definition, $S^{(m)}_{-,x} \subset \mathrm{Q}$, $x \in \mathrm{Q}$, is, as before, the support $supp\mu_{\mathcal{P}_{x,-}} := S^{(m)}_{-,x} \in \mathcal{K}(\mathrm{Q})$ of the corresponding to the operator (3.54) finite Borel measure $\mu_{\mathcal{P}_{x,-}}$ defined on the Borel subsets of $\mathrm{Q}\subset \mathbb{R}^m$.

It is naturally to put now $x \in \partial S^{(m)}_{+,x} \cap \partial S^{(m)}_{-,x}$, being an intrinsic point of the boundary $\partial S^{(m)}_{+,x}\backslash\partial Q = -\partial S^{(m)}_{-,x}\backslash\partial Q := \sigma^{(m-1)}_x \in \mathcal{K}(\mathrm{Q})$, where $\mathcal{K}(\mathrm{Q})$ is, as before, some singular simplicial complex generated by the open set $\mathrm{Q} \subset \mathbb{R}^m$. Thus, for our Fredholm operator $\mathbf{\Omega} := 1 + \Phi \in \mathrm{V}_f$ the corresponding factorization is written down as

$$\mathbf{\Omega} = (1 + \mathrm{K}_+(\mathbf{\Omega}))^{-1}(1 + \mathrm{K}_-(\mathbf{\Omega})) := \mathbf{\Omega}^{-1}_+\mathbf{\Omega}_-, \tag{3.55}$$

where integral operators $\mathrm{K}_\pm(\mathbf{\Omega}) \in \mathcal{B}^\pm_\infty(\mathcal{H})$ are given by expression (3.52) and (3.54) parametrized by a running intrinsic point $x \in \mathrm{Q}$.

4. The differential-geometric structure of a Lagrangian identity and related Delsarte transmutation operators

4.1. In Chapter 3 above we have studied in detail the spectral structure of Delsarte transmutation Volterrian operators $\mathbf{\Omega}_\pm \in Aut(\mathcal{H}) \cap \mathcal{B}(\mathcal{H})$ factorizing some Fredholm operator $\mathbf{\Omega} = \mathbf{\Omega}^{-1}_+\mathbf{\Omega}_-$ and stated their relationships with the approach suggested in [18, 19]. In particular, we demonstrated the existence of some Borel measures $\mu_{\mathcal{P}_{x,\pm}}$ localized upon hypersurfaces $S^{(m)}_{\pm,x} \in \mathcal{K}(\mathrm{Q})$ and related naturally with the corresponding integral operators $\mathrm{K}_\pm(\mathbf{\Omega})$, whose kernels $\hat{\mathrm{K}}_\pm(\mathbf{\Omega}) \in \mathcal{H}_-\otimes\mathcal{H}_-$ are congruent to a pair of given differential operators $(L, \tilde{L}) \subset \mathcal{L}(\mathcal{H})$, satisfying the relationships (3.12). In what will follow below we shall study some differential-geometric properties of the Lagrange identity naturally associated with two Delsarte related differential operators L and $\tilde{L}$ in $\mathcal{H}$ and describe by means of some specially constructed integral operator kernels the corresponding Delsarte transmutation operators exactly in the same spectral form as it was studied in Chapter 3 above.

Let a multi-dimensional linear differential operator $\mathrm{L} : \mathcal{H} \to \mathcal{H}$ of order $n(\mathrm{L}) \in \mathbb{Z}_+$ be of the form

$$\mathrm{L}(x|\partial) := \sum_{|\alpha|=0}^{n(\mathrm{L})} a_\alpha(x)\frac{\partial^{|\alpha|}}{\partial x^\alpha}, \tag{4.1}$$

and defined on a dense domain $D(\mathrm{L}) \subset \mathcal{H}$, where, as usually, $\alpha \in \mathbb{Z}^m_+$ is a multi-index, $x \in \mathbb{R}^m$, and for brevity one assumes that coefficients $a_\alpha \in \mathcal{S}(\mathbb{R}^m; End\mathbb{C}^N)$,

$\alpha \in \mathbb{Z}_+^m$. Consider the following easily derivable generalized Lagrangian identity for the differential expression (4.1) :

$$< \mathrm{L}^*\varphi, \psi > - < \varphi, \mathrm{L}\psi >= \sum_{i=1}^{m} (-1)^{i+1} \frac{\partial}{\partial x_i} Z_i[\varphi, \psi], \tag{4.2}$$

where $(\varphi, \psi) \in \mathcal{H}^* \times \mathcal{H}$, mappings $Z_i : \mathcal{H}^* \times \mathcal{H} \to \mathbb{C}$, $i = \overline{1, m}$, are semilinear due to the construction and $\mathrm{L}^* : \mathcal{H}^* \to \mathcal{H}^*$ is the corresponding formally conjugated to (4.1) differential expression, that is

$$\mathrm{L}^*(x|\partial) := \sum_{|\alpha|=0}^{n(\mathcal{L})} (-1)^{|\alpha|} \frac{\partial^{|\alpha|}}{\partial x^\alpha} \cdot \bar{a}_\alpha^{\intercal}(x).$$

Having multiplied the identity (4.2) by the usual oriented Lebesgue measure $dx = \wedge_{j=\overrightarrow{1,m}} dx_j$, we get that

$$< \mathrm{L}^*\varphi, \psi > dx - < \varphi, \mathrm{L}\psi > dx = dZ^{(m-1)}[\varphi, \psi] \tag{4.3}$$

for all $(\varphi, \psi) \in \mathcal{H}^* \times \mathcal{H}$, where

$$Z^{(m-1)}[\varphi, \psi] := \sum_{i=1}^{m} dx_1 \wedge dx_2 \wedge ... \wedge dx_{i-1} \wedge Z_i[\varphi, \psi] dx_{i+1} \wedge ... \wedge dx_m \tag{4.4}$$

is an $(m-1)-$differential form on $\mathbb{R}^m$.

4.2. Consider now all such pairs $(\varphi(\lambda), \psi(\mu)) \in \mathcal{H}_0^* \times \mathcal{H}_0 \subset \mathcal{H}_- \times \mathcal{H}_-$, $\lambda, \mu \in \Sigma$, where as before

$$\mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_- \tag{4.5}$$

is the usual Gelfand triple of Hilbert spaces [8, 9] related with our Hilbert-Schmidt rigged Hilbert space $\mathcal{H}$, $\Sigma \in \mathbb{C}^p$, $p \in \mathbb{Z}_+$, is some fixed measurable space of parameters endowed with a finite Borel measure ρ, that the differential form (4.4) is exact, that is there exists a set of $(m-2)-$differential forms $\Omega^{(m-2)}[\varphi(\lambda), \psi(\mu)] \in \Lambda^{m-2}(\mathbb{R}^m; \mathbb{C})$, $\lambda, \mu \in \Sigma$, on $\mathbb{R}^m$ satisfying the condition

$$Z^{(m-1)}[\varphi(\lambda), \psi(\mu)] = d\Omega^{(m-2)}[\varphi(\lambda), \psi(\mu)]. \tag{4.6}$$

A way to realize this condition is to take some closed subspaces $\mathcal{H}_0^*$ and $\mathcal{H}_0 \subset \mathcal{H}_-$ as solutions to the corresponding linear differential equations under some boundary conditions:

$$\begin{aligned} \mathcal{H}_0 \quad &: \quad = \{\psi(\lambda) \in \mathcal{H}_- : \mathrm{L}\psi(\lambda) = 0, \quad \psi(\lambda)|_{x\in\Gamma} = 0, \ \lambda \in \Sigma\}, \\ \mathcal{H}_0^* \quad &: \quad = \{\varphi(\lambda) \in \mathcal{H}_-^* : \mathrm{L}^*\varphi(\lambda) = 0, \quad \varphi(\lambda)|_{x\in\Gamma} = 0, \ \lambda \in \Sigma\}. \end{aligned}$$

The triple (4.5) allows naturally to determine properly a set of generalized eigenfunctions for extended operators L ,$\mathrm{L}^* : \mathcal{H}_- \to \mathcal{H}_-$, if $\Gamma \subset \mathbb{R}^m$ is taken as some (n-1)-dimensional piece-wise smooth hypersurface embedded into the configuration space $\mathbb{R}^m$. There can exist, evidently, situations [13, 7, 5] when boundary conditions are not necessary.

Let now $S_\pm(\sigma_x^{(m-2)}, \sigma_{x_0}^{(m-2)}) \in H_{m-1}(M; \mathbb{C})$ denote some two non-intersecting $(m-1)$-dimensional piece-wise smooth hypersurfaces from the homology group $H_{m-1}(M; \mathbb{C})$ of some topological compactification $M := \bar{\mathbb{R}}^m$, such that their boundaries are the same, that is $\partial S_\pm(\sigma_x^{(m-2)}, \sigma_{x_0}^{(m-2)}) = \sigma_x^{(m-2)} - \sigma_{x_0}^{(m-2)}$ and, additionally, $\partial(S_+(\sigma_x^{(m-2)}, \sigma_{x_0}^{(m-2)}) \cup S_-(\sigma_x^{(m-2)}, \sigma_{x_0}^{(m-2)})) = \oslash$, where $\sigma_x^{(m-2)}$ and

$\sigma_{x_0}^{(m-2)} \in C_{m-2}(\mathbb{R}^m;\mathbb{C})$ are some $(m-2)$-dimensional homological cycles from a suitable chain complex $\mathcal{K}(M)$ parametrized formally by means of two points $x, x_0 \in M$ and related in some way with the chosen above hypersurface $\Gamma \subset M$. Then from (4.6) based on the general Stokes theorem [26, 27, 30, 28] one correspondingly gets easily that

$$\int_{S_\pm(\sigma_x^{(m-2)},\sigma_{x_0}^{(m-2)})} Z^{(m-1)}[\varphi(\lambda),\psi(\mu)] = \int_{\partial S_\pm(\sigma_x^{(m-2)},\sigma_{x_0}^{(m-2)})} \Omega^{(m-2)}[\varphi(\lambda),\psi(\mu)] =$$

$$\int_{\sigma_x^{(m-2)}} \Omega^{(m-2)}[\varphi(\lambda),\psi(\mu)] - \int_{\sigma_{x_0}^{(m-2)}} \Omega^{(m-2)}[\varphi(\lambda),\psi(\mu)] \tag{4.7}$$
$$: \ = \Omega_x(\lambda,\mu) - \Omega_{x_0}(\lambda,\mu),$$

$$\int_{S_\pm(\sigma_x^{(m-2)},\sigma_{x_0}^{(m-2)})} \bar{Z}^{(m-1),\intercal}[\varphi(\lambda),\psi(\mu)] = \int_{\partial S_\pm(\sigma_x^{(m-2)},\sigma_{x_0}^{(m-2)})} \bar{\Omega}^{(m-2),\intercal}[\varphi(\lambda),\psi(\mu)] =$$

$$\int_{\sigma_x^{(m-2)}} \bar{\Omega}^{(m-2),\intercal}[\varphi(\lambda),\psi(\mu)] - \int_{\sigma_{x_0}^{(m-2)}} \bar{\Omega}^{(m-2),\intercal}[\varphi(\lambda),\psi(\mu)]$$
$$: \ = \Omega_x^{\circledast}(\lambda,\mu) - \Omega_{x_0}^{\circledast}(\lambda,\mu)$$

for the set of functions $(\varphi(\lambda),\psi(\mu)) \in \mathcal{H}_0^* \times \mathcal{H}_0,\ \lambda,\mu \in \Sigma$, with operator kernels $\Omega_x(\lambda,\mu)$, $\Omega_x^{\circledast}(\lambda,\mu)$ and $\Omega_{x_0}(\lambda,\mu)$, $\Omega_x^{\circledast}(\lambda,\mu)$, $\lambda,\mu \in \Sigma$, acting naturally in the Hilbert space $L_2^{(\rho)}(\Sigma;\mathbb{C})$.These kernels are assumed further to be nondegenerate in $L_2^{(\rho)}(\Sigma;\mathbb{C})$ and satisfying the homotopy conditions

$$\lim_{x\to x_0} \Omega_x(\lambda,\mu) \ = \ \Omega_{x_0}(\lambda,\mu), \quad \lim_{x\to x_0} \Omega_x^{\circledast}(\lambda,\mu) = \ \Omega_{x_0}^{\circledast}(\lambda,\mu).$$

4.3. Define now actions of the following two linear Delsarte permutations operators $\mathbf{\Omega}_\pm : \mathcal{H} \to \mathcal{H}$ and $\mathbf{\Omega}_\pm^{\circledast} : \mathcal{H}^* \to \mathcal{H}^*$ still upon a fixed set of functions $(\varphi(\lambda),\psi(\mu)) \in \mathcal{H}_0^* \times \mathcal{H}_0,\ \lambda,\mu \in \Sigma:$

$$\tilde{\psi}(\lambda) = \mathbf{\Omega}_\pm(\psi(\lambda)) := \int\limits_\Sigma d\rho(\eta) \int\limits_\Sigma d\rho(\mu)\psi(\eta)\Omega_x^{-1}(\eta,\mu)\Omega_{x_0}(\mu,\lambda),$$

$$\tilde{\varphi}(\lambda) = \mathbf{\Omega}_\pm^{\circledast}(\varphi(\lambda)) := \int\limits_\Sigma d\rho(\eta) \int\limits_\Sigma d\rho(\mu)\varphi(\eta)\Omega_x^{\circledast,-1}(\mu,\eta)\Omega_{x_0}^{\circledast}(\lambda,\mu). \tag{4.8}$$

Making use of the expressions (4.8), based on arbitrariness of the chosen set of functions $(\varphi(\lambda),\psi(\mu)) \in \mathcal{H}_0^* \times \mathcal{H}_0$, $\lambda,\mu \in \Sigma$, we can easily retrieve the corresponding operator expressions for operators $\mathbf{\Omega}_\pm$ and $\mathbf{\Omega}_\pm^{\circledast} : \mathcal{H}_- \to \mathcal{H}_-$, forcing the kernels $\Omega_{x_0}(\lambda,\mu)$ and $\Omega_{x_0}^{\circledast}(\lambda,\mu)$, $\lambda,\mu \in \Sigma$, to variate:

$$\begin{aligned}\tilde{\psi}(\lambda) \ &= \ \int\limits_\Sigma d\rho(\eta) \int\limits_\Sigma d\rho(\mu)\psi(\eta)\Omega_x(\eta,\mu)\Omega_x^{-1}(\mu,\lambda) \\ &- \int\limits_\Sigma d\rho(\eta) \int\limits_\Sigma d\rho(\mu)\psi(\eta)\Omega_x^{-1}(\eta,\mu)] \times \\ &\times \int_{S_\pm(\sigma_x^{(m-2)},\sigma_{x_0}^{(m-2)})} Z^{(m-1)}[\varphi(\mu),\psi(\lambda)])\end{aligned}$$

$$= \psi(\lambda) - \int_\Sigma d\rho(\eta)\int_\Sigma d\rho(\mu)\int_\Sigma d\rho(\nu)\int_\Sigma d\rho(\xi)\psi(\eta)\Omega_x^{-1}(\eta,\nu)\times$$

$$\times\Omega_{x_0}(\nu,\xi)]\Omega_{x_0}^{-1}(\xi,\mu)\int_{S_\pm(\sigma_x^{(m-2)},\sigma_{x_0}^{(m-2)})} Z^{(m-1)}[\varphi(\mu),\psi(\lambda)]$$

$$= \psi(\lambda) - \int_\Sigma d\rho(\eta)\int_\Sigma d\rho(\mu)\tilde\psi(\eta)\Omega_{x_0}^{-1}(\eta,\mu)]\int_{S_\pm(\sigma_x^{(m-2)},\sigma_{x_0}^{(m-2)})} Z^{(m-1)}[\varphi(\mu),\psi(\lambda)]$$

$$= (\mathbf{1} - \int_\Sigma d\rho(\eta)\int_\Sigma d\rho(\mu)\tilde\psi(\eta)\Omega_{x_0}^{-1}(\eta,\mu)\times$$

$$\times\int_{S_\pm(\sigma_x^{(m-2)},\sigma_{x_0}^{(m-2)})} Z^{(m-1)}[\varphi(\mu),(\cdot)])\ \psi(\lambda) := \mathbf{\Omega}_\pm\cdot\psi(\lambda);$$

$$\begin{aligned}\tilde\varphi(\lambda) \quad = \quad & \int_\Sigma d\rho(\eta)\int_\Sigma d\rho(\mu)\varphi(\eta)\Omega_x^{\circledast,-1}(\mu,\eta)\Omega_x^{\circledast}(\lambda,\mu)\\ & -\int_\Sigma d\rho(\eta)\int_\Sigma d\rho(\mu)\varphi(\eta)\Omega_x^{\circledast,-1}(\mu,\eta)\int_{S_\pm(\sigma_x^{(m-2)},\sigma_{x_0}^{(m-2)})}\bar Z^{(m-1),\intercal}[\varphi(\lambda),\psi(\mu)]\end{aligned}$$

$$= \varphi(\lambda) - \int_\Sigma d\rho(\eta)\int_\Sigma d\rho(\nu)\int_\Sigma d\rho(\xi)\int_\Sigma d\rho(\mu)\varphi(\eta)\Omega_x^{\circledast,-1}(\xi,\eta)\times$$

$$\times\Omega_{x_0}^{\circledast}(\nu,\xi)\Omega_{x_0}^{\circledast,-1}(\mu,\nu)\int_{S_\pm(\sigma_x^{(m-2)},\sigma_{x_0}^{(m-2)})}\bar Z^{(m-1),\intercal}[\varphi(\lambda),\psi(\mu)]$$

$$= (\mathbf{1} - \int_\Sigma d\rho(\eta)\int_\Sigma d\rho(\mu)\tilde\varphi(\eta)\Omega_{x_0}^{\circledast,-1}(\mu,\eta)\times \tag{4.9}$$

$$\times\int_{S_\pm(\sigma_x^{(m-2)},\sigma_{x_0}^{(m-2)})}\bar Z^{(m-1),\intercal}[(\cdot),\psi(\mu)])\ \varphi(\lambda) := \mathbf{\Omega}_\pm^{\circledast}\cdot\varphi(\lambda),$$

where, by definition,

$$\mathbf{\Omega}_\pm := \mathbf{1} - \int_\Sigma d\rho(\eta)\int_\Sigma d\rho(\mu)\tilde\psi(\eta)\Omega_{x_0}^{-1}(\eta,\mu)\int_{S_\pm(\sigma_x^{(m-2)},\sigma_{x_0}^{(m-2)})} Z^{(m-1)}[\varphi(\mu),(\cdot)]$$

$$\mathbf{\Omega}_\pm^{\circledast} := \mathbf{1} - \int_\Sigma d\rho(\eta)\int_\Sigma d\rho(\mu)\tilde\varphi(\eta)\Omega_{x_0}^{\circledast,-1}(\mu,\eta)\int_{S_\pm(\sigma_x^{(m-2)},\sigma_{x_0}^{(m-2)})}\bar Z^{(m-1),\intercal}[(\cdot),\psi(\mu)] \tag{4.10}$$

are of Volterra type multidimensional integral operators. It is to be noted here that now elements $(\varphi(\lambda),\psi(\mu))\in\mathcal{H}_0^*\times\mathcal{H}_0$ and $(\tilde\varphi(\lambda),\tilde\psi(\mu))\in\tilde{\mathcal{H}}_0^*\times\tilde{\mathcal{H}}_0$, $\lambda,\mu\in\Sigma$, inside the operator expressions (4.10) are not arbitrary but now fixed. Therefore, the operators (4.10) realize an extension of their actions (4.8) on a fixed pair of functions $(\varphi(\lambda),\psi(\mu))\in\mathcal{H}_0^*\times\mathcal{H}_0$, $\lambda,\mu\in\Sigma$, upon the whole functional space $\mathcal{H}^*\times\mathcal{H}$.

4.4. Due to the symmetry of expressions (4.8) and (4.10) with respect to two sets of functions $(\varphi(\lambda),\psi(\mu))\in\mathcal{H}_0^*\times\mathcal{H}_0$ and $(\tilde\varphi(\lambda),\tilde\psi(\mu))\in\tilde{\mathcal{H}}_0^*\times\tilde{\mathcal{H}}_0$, $\lambda,\mu\in\Sigma$, it is very easy to state the following lemma.

Lemma 4.1. *Operators (4.10) are bounded and invertible of Volterra type expressions in $\mathcal{H}^* \times \mathcal{H}$ whose inverse are given as follows:*

(4.11)

$$\mathbf{\Omega}_{\pm}^{-1} := \mathbf{1} - \int_{\Sigma} d\rho(\eta) \int_{\Sigma} d\rho(\mu)\psi(\eta)\tilde{\Omega}_{x_0}^{-1}(\eta,\mu) \int_{S_{\pm}(\sigma_x^{(m-2)},\sigma_{x_0}^{(m-2)})} Z^{(m-1)}[\tilde{\varphi}(\mu),(\cdot)]$$

$$\mathbf{\Omega}_{\pm}^{\circledast,-1} := \mathbf{1} - \int_{\Sigma} d\rho(\eta) \int_{\Sigma} d\rho(\mu)\varphi(\eta)\Omega_{x_0}^{\circledast,-1}(\mu,\eta) \int_{S_{\pm}(\sigma_x^{(m-2)},\sigma_{x_0}^{(m-2)})} \bar{Z}^{(m-1),\intercal}[(\cdot),\tilde{\psi}(\mu)]$$

where two sets of functions $(\varphi(\lambda),\psi(\mu)) \in \mathcal{H}_0^* \times \mathcal{H}_0$ and $(\tilde{\varphi}(\lambda),\tilde{\psi}(\mu)) \in \tilde{\mathcal{H}}_0^* \times \tilde{\mathcal{H}}_0$, $\lambda,\mu \in \Sigma$, are taken arbitrary but fixed.

For the expressions (4.11) to be compatible with mappings (4.8) the following actions must hold:

$$\psi(\lambda) = \mathbf{\Omega}_{\pm}^{-1} \cdot \tilde{\psi}(\lambda) = \int_{\Sigma} d\rho(\eta) \int_{\Sigma} d\rho(\mu)\tilde{\psi}(\eta)\tilde{\Omega}_x^{-1}(\eta,\mu)]\tilde{\Omega}_{x_0}(\mu,\lambda),$$

$$\varphi(\lambda) = \mathbf{\Omega}_{\pm}^{\circledast,-1} \cdot \tilde{\varphi}(\lambda) = \int_{\Sigma} d\rho(\eta) \int_{\Sigma} d\rho(\mu)\tilde{\varphi}(\eta)\tilde{\Omega}_x^{\circledast,-1}(\mu,\eta)\tilde{\Omega}_{x_0}^{\circledast}(\lambda,\mu), \tag{4.12}$$

where for any two sets of functions $(\varphi(\lambda),\psi(\mu)) \in \mathcal{H}_0^* \times \mathcal{H}_0$ and $(\tilde{\varphi}(\lambda),\tilde{\psi}(\mu)) \in \tilde{\mathcal{H}}_0^* \times \tilde{\mathcal{H}}_0$, $\lambda,\mu \in \Sigma$, the next relationship is satisfied:

$$(<\tilde{\mathrm{L}}^*\tilde{\varphi}(\lambda),\tilde{\psi}(\mu)> - <\tilde{\varphi}(\lambda),\tilde{\mathrm{L}}\tilde{\psi}(\mu)>)dx = d(\tilde{Z}^{(m-1)}[\tilde{\varphi}(\lambda),\tilde{\psi}(\mu)]),$$

$$\tilde{Z}^{(m-1)}[\tilde{\varphi}(\lambda),\tilde{\psi}(\mu)] = d\tilde{\Omega}^{(m-2)}[\tilde{\varphi}(\lambda),\tilde{\psi}(\mu)] \tag{4.13}$$

when

$$\tilde{\mathrm{L}} := \mathbf{\Omega}_{\pm}\mathrm{L}\mathbf{\Omega}_{\pm}^{-1}, \quad \tilde{\mathrm{L}}^* := \mathbf{\Omega}_{\pm}^{\circledast}\mathrm{L}^*\mathbf{\Omega}_{\pm}^{\circledast,-1}, \tag{4.14}$$

Moreover, the expressions above for $\tilde{\mathrm{L}} : \mathcal{H} \to \mathcal{H}$ and $\tilde{\mathrm{L}}^* : \mathcal{H}^* \to \mathcal{H}^*$ don't depend on the choice of the indexes below of operators $\mathbf{\Omega}_+$ or $\mathbf{\Omega}_-$ and are in the result differential. Since the last condition determines properly Delsarte transmutation operators (4.11), we need to state the following theorem.

Theorem 4.2. *The pair $(\tilde{\mathrm{L}}\ \tilde{\mathrm{L}}^*)$ of operator expressions $\tilde{\mathrm{L}} := \mathbf{\Omega}_{\pm}\mathrm{L}\mathbf{\Omega}_{\pm}^{-1}$ and $\tilde{\mathrm{L}}^* := \mathbf{\Omega}_{\pm}^{\circledast}\mathrm{L}^*\mathbf{\Omega}_{\pm}^{\circledast,-1}$ acting in the space $\mathcal{H} \times \mathcal{H}^*$ is purely differential for any suitably chosen hyper-surfaces $S_{\pm}(\sigma_x^{(m-2)},\sigma_{x_0}^{(m-2)}) \in H_{m-1}(M;\mathbb{C})$ from the homology group $H_{m-1}(M;\mathbb{C})$.*

Proof. For proving the theorem it is necessary to show that the formal pseudo-differential expressions corresponding to operators $\tilde{\mathrm{L}}$ and $\tilde{\mathrm{L}}^*$ contain no integral elements. Making use of an idea devised in [23, 17], one can formulate such a lemma. ∎

Lemma 4.3. *A pseudo-differential operator $\mathrm{L} : \mathcal{H} \to \mathcal{H}$ is purely differential iff the following equality*

$$(h, (\mathrm{L}\frac{\partial^{|\alpha|}}{\partial x^{\alpha}})_+ f) = (h, \mathrm{L}_+\frac{\partial^{|\alpha|}}{\partial x^{\alpha}} f) \tag{4.15}$$

holds for any $|\alpha| \in \mathbb{Z}_+$ and all $(h,f) \in \mathcal{H}^* \times \mathcal{H}$, that is the condition (4.15) is equivalent to the equality $\mathrm{L}_+ = \mathrm{L}$, where, as usually, the sign "$(...)_+$" means the purely differential part of the corresponding expression inside the bracket.

Based now on this Lemma and exact expressions of operators (4.10), similarly to calculations done in [23], one shows right away that operators $\tilde{\mathrm{L}}$ and $\tilde{\mathrm{L}}^*$, depending correspondingly only both on the homological cycles $\sigma_x^{(m-2)}, \sigma_{x_0}^{(m-2)} \in C_{m-2}(M;\mathbb{C})$ from a simplicial chain complex $\mathcal{K}(M)$, marked by points $x, x_0 \in \mathbb{R}^m$, and on two sets of functions $(\varphi(\lambda), \psi(\mu)) \in \mathcal{H}_0^* \times \mathcal{H}_0$ and $(\tilde{\varphi}(\lambda), \tilde{\psi}(\mu)) \in \tilde{\mathcal{H}}_0^* \times \tilde{\mathcal{H}}_0$, $\lambda, \mu \in \Sigma$, are purely differential thereby finishing the proof.►

The differential-geometric construction suggested above can be nontrivially generalized for the case of $m \in \mathbb{Z}_+$ commuting to each other differential operators in a Hilbert space $\mathcal{H}$ giving rise to a new look at theory of Delsarte transmutation operators based on differential-geometric and topological de Rham-Hodge techniques. These aspects will be discussed in detail in the next two chapters below.

5. The general differential-geometric and topological structure of Delsarte transmutation operators: the De Rham-Hodge-Skrypnik theory

5.1. Below we shall explain the corresponding differential-geometric and topological nature of these spectral related results obtained above and generalize them to a set $\mathcal{L}$ of commuting differential operators Delsarte related with another commuting set $\tilde{\mathcal{L}}$ of differential operators in $\mathcal{H}$. These results are deeply based on the De Rham-Hodge-Skrypnik theory [28, 29, 27, 3, 30] of special differential complexes giving rise to effective analytical expressions for the corresponding Delsarte transmutation Volterra type operators in a given Hilbert space $\mathcal{H}$. As a by-product one obtains the integral operator structure of Delsarte transmutation operators for polynomial pencils of differential operators in $\mathcal{H}$ having many applications both in spectral theory of such multidimensional operator pencils [21, 22, 24, 33] and in soliton theory [17, 12, 6, 25, 11] of multidimensional integrable dynamical systems on functional manifolds, being very important for diverse applications in modern mathematical physics.

Let $M := \bar{\mathbb{R}}^m$ denote as before a suitably compactified metric space of dimension $m = dim M \in \mathbb{Z}_+$ (without boundary) and define some finite set $\mathcal{L}$ of smooth commuting to each other linear differential operators

$$\mathrm{L}_j(x|\partial) := \sum_{|\alpha|=0}^{n(L_j)} a_\alpha^{(j)}(x)\partial^{|\alpha|}/\partial x^\alpha \tag{5.1}$$

with respect to $x \in M$, having Schwatrz coefficients $a_\alpha^{(j)} \in \mathcal{S}(M; End\mathbb{C}^N)$, $|\alpha| = \overline{0, n(\mathrm{L}_j)}$, $n(\mathrm{L}_j) \in \mathbb{Z}_+$, $j = \overline{1,m}$, and acting in the Hilbert space $\mathcal{H} := L_2(M;\mathbb{C}^N)$. It is assumed also that domains $D(\mathrm{L}_j) := D(\mathcal{L}) \subset \mathcal{H}$, $j = \overline{1,m}$, are dense in $\mathcal{H}$.

Consider now a generalized external anti-differentiation operator $d_{\mathcal{L}}$ $:\Lambda(M;\mathcal{H}) \rightarrow \Lambda(M;\mathcal{H})$ acting in the Grassmann algebra $\Lambda(M;\mathcal{H})$ as follows: for any $\beta^{(k)} \in \Lambda^k(M;\mathcal{H})$, $k = \overline{0,m}$,

$$d_{\mathcal{L}}\beta^{(k)} := \sum_{j=1}^{m} dx_j \wedge \mathrm{L}_j(x;\partial)\beta^{(k)} \in \Lambda^{k+1}(M;\mathcal{H}). \tag{5.2}$$

It is easy to see that the operation (5.2) in the case $\mathrm{L}_j(x;\partial) := \partial/\partial x_j$, $j = \overline{1,m}$, coincides exactly with the standard external differentiation $d = \sum_{j=1}^m dx_j \wedge \partial/\partial x_j$

on the Grassmann algebra $\Lambda(M;\mathcal{H})$. Making use of the operation (5.2) on $\Lambda(M;\mathcal{H})$, one can construct the following generalized de Rham co-chain complex

$$\mathcal{H} \to \Lambda^0(M;\mathcal{H}) \overset{d_\mathcal{L}}{\to} \Lambda^1(M;\mathcal{H}) \overset{d_\mathcal{L}}{\to} ... \overset{d_\mathcal{L}}{\to} \Lambda^m(M;\mathcal{H}) \overset{d_\mathcal{L}}{\to} 0. \tag{5.3}$$

The following important property concerning the complex (5.3) holds.

Lemma 5.1. *The co-chain complex (5.3) is exact.*

Proof. It follows easily from the equality $d_\mathcal{L}d_\mathcal{L} = 0$ holding due to the commutation of operators (5.1) .▷ ∎

5.2. Below we will follow the ideas developed before in [3, 29]. A differential form $\beta \in \Lambda(M;\mathcal{H})$ will be called $d_\mathcal{L}$-closed if $d_\mathcal{L}\beta = 0$, and a form $\gamma \in \Lambda(M;\mathcal{H})$ will be called $d_\mathcal{L}$-homological to zero if there exists on M such a form $\omega \in \Lambda(M;\mathcal{H})$ that $\gamma = d_\mathcal{L}\omega$.

Consider now the standard algebraic Hodge star-operation

$$\star : \Lambda^k(M;\mathcal{H}) \to \Lambda^{m-k}(M;\mathcal{H}), \tag{5.4}$$

$k = \overline{0,m}$, as follows [4, 28, 29, 30]: if $\beta \in \Lambda^k(M;\mathcal{H})$, then the form $\star\beta \in \Lambda^{m-k}(M;\mathcal{H})$ is such that:

i) $(m-k)$-dimensional volume $|\star\beta|$ of the form $\star\beta$ equals k-dimensional volume $|\beta|$ of the form β;

ii) the m-dimensional measure $\bar{\beta}^\intercal \wedge \star\beta > 0$ under the fixed orientation on M.

Define also on the space $\Lambda(M;\mathcal{H})$ the following natural scalar product: for any $\beta, \gamma \in \Lambda^k(M;\mathcal{H})$, $k = \overline{0,m}$,

$$(\beta, \gamma) := \int_M \bar{\beta}^\intercal \wedge \star\gamma. \tag{5.5}$$

Subject to the scalar product (5.5) we can naturally construct the corresponding Hilbert space

$$\mathcal{H}_\Lambda(M) := \overset{m}{\underset{k=0}{\oplus}} \mathcal{H}^k_\Lambda(M)$$

well suitable for our further consideration. Notice also here that the Hodge star $\star$-operation satisfies the following easily checkable property: for any $\beta, \gamma \in \mathcal{H}^k_\Lambda(M)$, $k = \overline{0,m}$,

$$(\beta, \gamma) = (\star\beta, \star\gamma), \tag{5.6}$$

that is the Hodge operation $\star : \mathcal{H}_\Lambda(M) \to \mathcal{H}_\Lambda(M)$ is isometry and its standard adjoint with respect to the scalar product (5.5) operation satisfies the condition $(\star)' = (\star)^{-1}$.

Denote by $d'_\mathcal{L}$ the formally adjoint expression to the external weak differential operation $d_\mathcal{L} : \mathcal{H}_\Lambda(M) \to \mathcal{H}_\Lambda(M)$ in the Hilbert space $\mathcal{H}_\Lambda(M)$. Making now use of the operations $d'_\mathcal{L}$ and $d_\mathcal{L}$ in $\mathcal{H}_\Lambda(M)$ one can naturally define [4, 29, 30] the generalized Laplace-Hodge operator $\Delta_\mathcal{L} : \mathcal{H}_\Lambda(M) \to \mathcal{H}_\Lambda(M)$ as

$$\Delta_\mathcal{L} := d'_\mathcal{L} d_\mathcal{L} + d_\mathcal{L} d'_\mathcal{L}. \tag{5.7}$$

Take a form $\beta \in \mathcal{H}_\Lambda(M)$ satisfying the equality

$$\Delta_\mathcal{L}\beta = 0. \tag{5.8}$$

Such a form is called *harmonic*. One can also verify that a harmonic form $\beta \in \mathcal{H}_\Lambda(M)$ satisfies simultaneously the following two adjoint conditions:

$$d'_{\mathcal{L}}\beta = 0, \quad d_{\mathcal{L}}\beta = 0, \tag{5.9}$$

easily stemming from (5.7) and (5.9).

It is not hard to check that the following differential operation in $\mathcal{H}_\Lambda(M)$

$$d^*_{\mathcal{L}} := \star d'_{\mathcal{L}}(\star)^{-1} \tag{5.10}$$

defines also a usual [26, 27, 28] external anti-differential operation in $\mathcal{H}_\Lambda(M)$. The corresponding dual to (5.3) co-chain complex

$$\mathcal{H} \to \Lambda^0(M;\mathcal{H}) \overset{d^*_{\mathcal{L}}}{\to} \Lambda^1(M;\mathcal{H}) \overset{d^*_{\mathcal{L}}}{\to} \ldots \overset{d^*_{\mathcal{L}}}{\to} \Lambda^m(M;\mathcal{H}) \overset{d^*_{\mathcal{L}}}{\to} 0 \tag{5.11}$$

is evidently exact too, as the property $d^*_{\mathcal{L}}d^*_{\mathcal{L}} = 0$ holds due to the definition (5.7).

5.3. Denote further by $\mathcal{H}^k_{\Lambda(\mathcal{L})}(M)$, $k = \overline{0,m}$, the cohomology groups of $d_{\mathcal{L}}$-closed and by $\mathcal{H}^k_{\Lambda(\mathcal{L}^*)}(M)$, $k = \overline{0,m}$, the cohomology groups of $d^*_{\mathcal{L}}$-closed differential forms, correspondingly, and by $\mathcal{H}^k_{\Lambda(\mathcal{L}^*\mathcal{L})}(M)$, $k = \overline{0,m}$, the abelian groups of harmonic differential forms from the Hilbert sub-spaces $\mathcal{H}^k_\Lambda(M)$, $k = \overline{0,m}$. Before formulating next results, define the standard Hilbert-Schmidt rigged chain [8] of positive and negative Hilbert spaces of differential forms

$$\mathcal{H}^k_{\Lambda,+}(M) \subset \mathcal{H}^k_\Lambda(M) \subset \mathcal{H}^k_{\Lambda,-}(M) \tag{5.12}$$

and the corresponding rigged chains of Hilbert sub-spaces for harmonic forms

$$\mathcal{H}^k_{\Lambda(\mathcal{L}^*\mathcal{L}),+}(M) \subset \mathcal{H}^k_{\Lambda(\mathcal{L}^*\mathcal{L})}(M) \subset \mathcal{H}^k_{\Lambda(\mathcal{L}^*\mathcal{L}),-}(M), \tag{5.13}$$

and cohomology groups:

$$\begin{aligned} \mathcal{H}^k_{\Lambda(\mathcal{L}),+}(M) &\subset \mathcal{H}^k_{\Lambda(\mathcal{L})}(M) \subset \mathcal{H}^k_{\Lambda(\mathcal{L}),-}(M), \\ \mathcal{H}^k_{\Lambda(\mathcal{L}^*),+}(M) &\subset \mathcal{H}^k_{\Lambda(\mathcal{L}^*)}(M) \subset \mathcal{H}^k_{\Lambda(\mathcal{L}^*),-}(M), \end{aligned} \tag{5.14}$$

for any $k = \overline{0,m}$. Assume also that the Laplace-Hodge operator (5.7) is elliptic in $\mathcal{H}^0_\Lambda(M)$. Now by reasonings similar to those in [4, 27, 28, 30] one can formulate the following a little generalized de Rham-Hodge theorem.

Theorem 5.2. *The groups of harmonic forms $\mathcal{H}^k_{\Lambda(\mathcal{L}^*\mathcal{L}),-}(M)$, $k = \overline{0,m}$, are, correspondingly, isomorphic to the cohomology groups $(H^k(M;\mathbb{C}))^\Sigma$, $k = \overline{0,m}$, where $H^k(M;\mathbb{C})$ is the $k-th$ cohomology group of the manifold M with complex coefficients, $\Sigma \subset \mathbb{C}^p$ is a set of suitable "spectral" parameters marking the linear space of independent $d^*_{\mathcal{L}}$-closed 0-forms from $\mathcal{H}^0_{\Lambda(\mathcal{L}),-}(M)$ and, moreover, the following direct sum decompositions*

$$\mathcal{H}^k_{\Lambda(\mathcal{L}^*\mathcal{L}),-}(M) \oplus \Delta\mathcal{H}^k_-(M) = \mathcal{H}^k_{\Lambda,-}(M) = \mathcal{H}^k_{\Lambda(\mathcal{L}^*\mathcal{L}),-}(M) \oplus d_{\mathcal{L}}\mathcal{H}^{k-1}_{\Lambda,-}(M) \oplus d'_{\mathcal{L}}\mathcal{H}^{k+1}_{\Lambda,-}(M) \tag{5.15}$$

hold for any $k = \overline{0,m}$.

Another variant of the statement similar to that above was formulated in [3] and reads as the following generalized de Rham-Hodge-Skrypnik theorem.

Theorem 5.3. *(See Skrypnik I.V.* [3]*) The generalized cohomology groups $\mathcal{H}^k_{\Lambda(\mathcal{L}),-}(M), k = \overline{0,m}$, are isomorphic, correspondingly, to the cohomology groups $(H^k(M;\mathbb{C}))^\Sigma$, $k = \overline{0,m}$.*

A proof of this theorem is based on some special sequence [3] of differential Lagrange type identities. Define the following closed subspace

$$\mathcal{H}_0^* := \{\varphi^{(0)}(\lambda) \in \mathcal{H}^0_{\Lambda(\mathcal{L}^*),-}(M) : d^*_{\mathcal{L}}\varphi^{(0)}(\lambda) = 0,\ \varphi^{(0)}(\lambda)|_\Gamma = 0,\ \lambda \in \Sigma\} \tag{5.16}$$

for some smooth $(m-1)$-dimensional hypersurface $\Gamma \subset M$ and $\Sigma \subset (\sigma(\mathcal{L}) \cap \bar{\sigma}(\mathcal{L}^*)) \times \Sigma_\sigma \subset \mathbb{C}^p$, where $\mathcal{H}^0_{\Lambda(\mathcal{L}^*),-}(M)$ is, as above, a suitable Hilbert-Schmidt rigged [8, 9] zero-order cohomology group Hilbert space from the chain given by (5.14), $\sigma(\mathcal{L})$ and $\sigma(\mathcal{L}^*)$ are, correspondingly, mutual spectra of the sets of commuting operators $\mathcal{L}$ and $\mathcal{L}^*$. Thereby the dimension $\dim\ \mathcal{H}_0^* = card\ \Sigma$ is assumed to be known.

The next lemma stated by Skrypnik I.V. [3] being fundamental for the proof holds.

Lemma 5.4. *(See Skrypnik I.V. [3]) There exists a set of differential $(k+1)$-forms $Z^{(k+1)}[\varphi^{(0)}(\lambda), d_{\mathcal{L}}\psi^{(k)}] \in \Lambda^{k+1}(M;\mathcal{H})$, $k = \overline{0,m}$, and a set of k-forms $Z^{(k)}[\varphi^{(0)}(\lambda), \psi^{(k)}] \in \Lambda^k(M;\mathcal{H})$, $k = \overline{0,m}$, parametrized by a set $\Sigma \ni \lambda$ and semilinear in $(\varphi^{(0)}(\lambda), \psi^{(k)}) \in \mathcal{H}_0^* \times \mathcal{H}^k_{\Lambda,-}(M)$, such that*

$$Z^{(k+1)}[\varphi^{(0)}(\lambda), d_{\mathcal{L}}\psi^{(k)}] = dZ^{(k)}[\varphi^{(0)}(\lambda), \psi^{(k)}] \tag{5.17}$$

for all $k = \overline{0,m}$ and $\lambda \in \Sigma$.

Proof. A proof is based on the following generalized Lagrange type identity holding for any pair $(\varphi^{(0)}(\lambda), \psi^{(k)}) \in \mathcal{H}_0^* \times \mathcal{H}^k_{\Lambda,-}(M)$:

$$\begin{aligned} 0 &= < d^*_{\mathcal{L}}\varphi^{(0)}(\lambda), \star(\psi^{(k)} \wedge \bar{\gamma}) > \\ &: =< \star d'_{\mathcal{L}}(\star)^{-1}\varphi^{(0)}(\lambda), \star(\psi^{(k)} \wedge \bar{\gamma}) > \\ &= < d'_{\mathcal{L}}(\star)^{-1}\varphi^{(0)}(\lambda), \psi^{(k)} \wedge \bar{\gamma} > = < (\star)^{-1}\varphi^{(0)}(\lambda), d_{\mathcal{L}}\psi^{(k)} \wedge \bar{\gamma} > \\ &\quad + Z^{(k+1)}[\varphi^{(0)}(\lambda), d_{\mathcal{L}}\psi^{(k)}] \wedge \bar{\gamma} \\ &= < (\star)^{-1}\varphi^{(0)}(\lambda), d_{\mathcal{L}}\psi^{(k)} \wedge \bar{\gamma} > + dZ^{(k)}[\varphi^{(0)}(\lambda), \psi^{(k)}] \wedge \bar{\gamma} \end{aligned} \tag{5.18}$$

where $Z^{(k+1)}[\varphi^{(0)}(\lambda), d_{\mathcal{L}}\psi^{(k)}] \in \Lambda^{k+1}(M;\mathbb{C})$, $k = \overline{0,m}$, and $Z^{(k)}[\varphi^{(0)}(\lambda), \psi^{(k)}] \in \Lambda^{k-1}(M;\mathbb{C})$, $k = \overline{0,m}$, are some semilinear differential forms parametrized by a parameter $\lambda \in \Sigma$, and $\bar{\gamma} \in \Lambda^{m-k-1}(M;\mathbb{C})$ is arbitrary constant $(m-k-1)$-form. Thereby, the semilinear differential k-forms $Z^{(k+1)}[\varphi^{(0)}(\lambda), d_{\mathcal{L}}\psi^{(k)}] \in \Lambda^{k+1}(M;\mathbb{C})$, $k = \overline{0,m}$, and k-forms $Z^{(k)}[\varphi^{(0)}(\lambda), \psi^{(k)}] \in \Lambda^k(M;\mathbb{C})$, $k = \overline{0,m}$, $\lambda \in \Sigma$, constructed above exactly constitute those searched for in the Lemma.$\triangleright$ ∎

Based now on this Lemma 3.3 one can construct the cohomology group isomorphism claimed in the Theorem 3.2 formulated above. Namely, following [3], let us take some singular simplicial [27, 28, 30] complex $\mathcal{K}(M)$ of the manifold M and introduce linear mappings $B^{(k)}_\lambda : \mathcal{H}^k_{\Lambda,-}(M) \to C_k(M;\mathbb{C}))$, $k = \overline{0,m}$, $\lambda \in \Sigma$, where $C_k(M;\mathbb{C})$, $k = \overline{0,m}$, are as before free abelian groups over the field $\mathbb{C}$ generated, correspondingly, by all k-chains of simplexes $S^{(k)} \in C_k(M;\mathbb{C})$, $k = \overline{0,m}$, from the singular simplicial complex $\mathcal{K}(M)$ as follows:

$$B^{(k)}_\lambda(\psi^{(k)}) := \sum_{S^{(k)} \in C_k(M;\mathbb{C}))} S^{(k)} \int_{S^{(k)}} Z^{(k)}[\varphi^{(0)}(\lambda), \psi^{(k)}] \tag{5.19}$$

with $\psi^{(k)} \in \mathcal{H}^k_\Lambda(M)$, $k = \overline{0,m}$. The following theorem based on mappings (5.19) holds.

Theorem 5.5. *(See Skrypnik I.V. [3]) The set of operations (5.19) parametrized by $\lambda \in \Sigma$ realizes the cohomology groups isomorphism formulated in the Theorem 3.3.*

Proof. A proof of this theorem one can get passing over in (5.19) to the corresponding cohomology $\mathcal{H}^k_{\Lambda(\mathcal{L}),-}(M)$ and homology $H_k(M;\mathbb{C})$ groups of M for every $k=\overline{0,m}$. If one to take an element $\psi^{(k)} := \psi^{(k)}(\mu) \in \mathcal{H}^k_{\Lambda(\mathcal{L}),-}(M)$, $k=\overline{0,m}$, solving the equation $d_{\mathcal{L}}\psi^{(k)}(\mu)=0$ with $\mu\in\Sigma_k$ being some set of the related "spectral" parameters marking elements of the subspace $\mathcal{H}^k_{\Lambda(\mathcal{L}),-}(M)$, then one finds easily from (5.19) and the identity (5.18) that

$$dZ^{(k)}[\varphi^{(0)}(\lambda),\psi^{(k)}(\mu)]=0 \tag{5.20}$$

for all pairs $(\lambda,\mu)\in\Sigma\times\Sigma_k$, $k=\overline{0,m}$. This, in particular, means due to the Poincare lemma [26, 27, 30, 28] that there exist differential $(k-1)$-forms $\Omega^{(k-1)}[\varphi^{(0)}(\lambda),\psi(\mu)]\in\Lambda^{k-1}(M;\mathbb{C})$, $k=\overline{0,m}$, such that

$$Z^{(k)}[\varphi^{(0)}(\lambda),\psi(\mu)] = d\Omega^{(k-1)}[\varphi^{(0)}(\lambda),\psi(\mu)] \tag{5.21}$$

for all pairs $(\varphi^{(0)}(\lambda),\psi^{(k)}(\mu))\in\mathcal{H}_0^*\times\mathcal{H}^k_{\Lambda(\mathcal{L}),-}(M)$ parametrized by $(\lambda,\mu)\in\Sigma\times\Sigma_k$, $k=\overline{0,m}$. As a result of passing on the right-hand side of (5.19) to the homology groups $H_k(M;\mathbb{C})$, $k=\overline{0,m}$, one gets due to the standard Stokes theorem [26, 30, 28, 27] that the mappings

$$B^{(k)}_\lambda : \mathcal{H}^k_{\Lambda(\mathcal{L}),-}(M) \rightleftarrows H_k(M;\mathbb{C}) \tag{5.22}$$

are isomorphisms for every $\lambda\in\Sigma$. Making further use of the Poincare duality [27, 28, 30] between the homology groups $H_k(M;\mathbb{C})$, $k=\overline{0,m}$, and the cohomology groups $H^k(M;\mathbb{C})$, $k=\overline{0,m}$, correspondingly, one obtains finally the statement claimed in theorem 3.5, that is $\mathcal{H}^k_{\Lambda(\mathcal{L}),-}(M)\simeq(H^k(M;\mathbb{C}))^{\Sigma}.\triangleright$ ∎

5.4. Assume now that $M := \mathrm{T}^r\times\bar{\mathbb{R}}^s$, $dimM = s+r\in\mathbb{Z}_+$, and $\mathcal{H} := L_2(T^r;L_2(\mathbb{R}^s;\mathbb{C}^N))$, where $T^r := \underset{j=1}{\overset{r}{\times}} T_j$, $T_j := [0,T_j)\subset\mathbb{R}_+$, $j=\overline{1,r}$, and put

$$d_{\mathcal{L}} = \sum_{j=1}^r dt_j\wedge \mathrm{L}_j(t;x|\partial),\ \mathrm{L}_j(t;x|\partial) := \partial/\partial t_j - L_j(t;x|\partial), \tag{5.23}$$

with

$$L_j(t;x|\partial) = \sum_{|\alpha|=0}^{n(L_j)} a^{(j)}_\alpha(t;x)\partial^{|\alpha|}/\partial x^\alpha, \tag{5.24}$$

$j=\overline{1,r}$, being differential operations parametrically dependent on $t\in T^r$ and defined on dense subspaces $D(\mathrm{L}_j)=D(\mathcal{L})\subset L_2(\mathbb{R}^s;\mathbb{C}^N)$, $j=\overline{1,r}$. It is assumed also that operators $\mathrm{L}_j:\mathcal{H}\to\mathcal{H}$, $j=\overline{1,r}$, are commuting to each other.

Take now such a fixed pair $(\varphi^{(0)}(\lambda),\psi^{(0)}(\mu)dx)\in\mathcal{H}_0^*\times\mathcal{H}^s_{\Lambda(\mathcal{L}),-}(M)$, parametrized by elements $(\lambda,\mu)\in\Sigma\times\Sigma$, for which due to both Theorem 6.5 and the Stokes theorem [26, 27, 30, 28] the equality

$$B^{(s)}_\lambda(\psi^{(0)}(\mu)dx) = S^{(s)}_{(t;x)}\int_{\partial S^{(s)}_{(t;x)}}\Omega^{(s-1)}[\varphi^{(0)}(\lambda),\psi^{(0)}(\mu)dx] \tag{5.25}$$

holds, where $S^{(s)}_{(t;x)}\in H_s(M;\mathbb{C})$ is some arbitrary but fixed element parametrized by an arbitrarily chosen point $(t;x)\in M\cap S^{(s)}_{(t;x)}$. Consider the next integral expressions

$$
\begin{aligned}
\Omega_{(t;x)}(\lambda,\mu) \quad &: \quad = \int_{\partial S^{(s)}_{(t;x)}} \Omega^{(s-1)}[\varphi^{(0)}(\lambda),\psi^{(0)}(\mu)dx],\\
\Omega_{(t_0;x_0)}(\lambda,\mu) \quad &: \quad = \int_{\partial S^{(s)}_{(t_0;x_0)}} \Omega^{(s-1)}[\varphi^{(0)}(\lambda),\psi^{(0)}(\mu)dx],
\end{aligned} \tag{5.26}
$$

where a point $(t_0;x_0)\in M\cap S^{(s)}_{(t_0;x_0)}$ is taken fixed, $\lambda,\mu\in\Sigma$, and interpret them as the corresponding kernels [8] of the integral invertible operators of Hilbert-Schmidt type $\Omega_{(t;x)},\Omega_{(t_0;x_0)}: L^{(\rho)}_2(\Sigma;\mathbb{C})\to L^{(\rho)}_2(\Sigma;\mathbb{C})$, where ρ is some finite Borel measure on the parameter set Σ. It assumes also above that the boundaries $\partial S^{(s)}_{(t;x)}:=\sigma^{(s-1)}_{(t;x)}$and $\partial S^{(s)}_{(t_0;x_0)}:=\sigma^{(s-1)}_{(t_0;x_0)}$ are taken homological to each other as $(t;x)\to(t_0;x_0)\in M$. Define now the expressions

$$
\mathbf{\Omega}_{\pm}:\psi^{(0)}(\eta)\to\tilde{\psi}^{(0)}(\eta) \tag{5.27}
$$

for $\psi^{(0)}(\eta)dx\in\mathcal{H}^s_{\Lambda(\mathcal{L}),-}(M)$ and some $\tilde{\psi}^{(0)}(\eta)dx\in\mathcal{H}^s_{\Lambda,-}(M)$, where by definition

$$
\begin{aligned}
\tilde{\psi}^{(0)}(\eta) &:= \psi^{(0)}(\eta)\cdot\Omega^{-1}_{(t;x)}\Omega_{(t_0;x_0)}\\
&= \int_\Sigma d\rho(\mu)\int_\Sigma d\rho(\xi)\psi^{(0)}(\mu)\Omega^{-1}_{(t;x)}(\mu,\xi)\Omega_{(t_0;x_0)}(\xi,\eta)
\end{aligned} \tag{5.28}
$$

for any $\eta\in\Sigma$ being motivated by the expression (5.25). Suppose now that the elements (5.28) are ones being related to some another Delsarte transformed cohomology group $\mathcal{H}^s_{\Lambda(\tilde{\mathcal{L}}),-}(M)$, that is the following condition

$$
d_{\tilde{\mathcal{L}}}\tilde{\psi}^{(0)}(\eta)dx=0 \tag{5.29}
$$

for $\tilde{\psi}^{(0)}(\eta)dx\in\mathcal{H}^s_{\Lambda(\tilde{\mathcal{L}}),-}(M)$, $\eta\in\Sigma$, and some new external anti-differentiation operation in $\mathcal{H}_{\Lambda,-}(M)$

$$
d_{\tilde{\mathcal{L}}}:=\sum_{j=1}^m dx_j\wedge\tilde{\mathrm{L}}_j(t;x|\partial),\ \tilde{\mathrm{L}}_j(t;x|\partial):=\partial/\partial t_j-\tilde{L}_j(t;x|\partial) \tag{5.30}
$$

holds, where expressions

$$
\tilde{L}_j(t;x|\partial)=\sum_{|\alpha|=0}^{n(L_j)}\tilde{a}^{(j)}_\alpha(t;x)\partial^{|\alpha|}/\partial x^\alpha, \tag{5.31}
$$

$j=\overline{1,r}$, are differential operations in $L_2(\mathbb{R}^s;\mathbb{C}^N)$ parametrically dependent on $t\in T^r$.

5.5. Put now that

$$
\tilde{\mathrm{L}}_j:=\mathbf{\Omega}_{\pm}\mathrm{L}_j\mathbf{\Omega}^{-1}_{\pm} \tag{5.32}
$$

for each $j=\overline{1,r}$, where $\mathbf{\Omega}_{\pm}:\mathcal{H}\to\mathcal{H}$ are the corresponding Delsarte transmutation operators related with some elements $S_{\pm}(\sigma^{(s-1)}_{(x;t)},\sigma^{(s-1)}_{(x_0;t_0)})\in H_s(M;\mathbb{C})$ related naturally with homological to each other boundaries $\partial S^{(s)}_{(x;t)}=\sigma^{(s-1)}_{(x;t)}$ and

$\partial S^{(s)}_{(x_0;t_0)} = \sigma^{(s-1)}_{(x_0;t_0)}$. Since all of operators $\mathrm{L}_j : \mathcal{H} \to \mathcal{H}$, $j = \overline{1,r}$, were taken commuting, the same property also holds for the transformed operators (5.32), that is $[\tilde{\mathrm{L}}_j, \tilde{\mathrm{L}}_k] = 0$, $k, j = \overline{0,m}$. The latter is, evidently, equivalent due to (5.32) to the following general expression:

$$d_{\tilde{\mathcal{L}}} = \mathbf{\Omega}_{\pm} d_{\mathcal{L}} \mathbf{\Omega}_{\pm}^{-1}. \tag{5.33}$$

For the condition (5.33) and (5.29) to be satisfied, let us consider the corresponding to (5.25) expressions

$$\tilde{B}^{(s)}_{\lambda}(\tilde{\psi}^{(0)}(\eta)dx) = S^{(s)}_{(t;x)} \tilde{\Omega}_{(t;x)}(\lambda, \eta), \tag{5.34}$$

related with the corresponding external differentiation (5.33), where $S^{(s)}_{(t;x)} \in H_s(M;\mathbb{C})$ and $(\lambda, \eta) \in \Sigma \times \Sigma$. Assume further that there are also defined mappings

$$\mathbf{\Omega}^{\circledast}_{\pm} : \varphi^{(0)}(\lambda) \to \tilde{\varphi}^{(0)}(\lambda) \tag{5.35}$$

with $\mathbf{\Omega}^{\circledast}_{\pm} : \mathcal{H}^* \to \mathcal{H}^*$ being some operators associated (but not necessary adjoint!) with the corresponding Delsarte transmutation operators $\mathbf{\Omega}_{\pm} : \mathcal{H} \to \mathcal{H}$ and satisfying the standard relationships $\tilde{\mathrm{L}}^*_j := \mathbf{\Omega}^{\circledast}_{\pm} \mathrm{L}^*_j \mathbf{\Omega}^{\circledast,-1}_{\pm}$, $j = \overline{1,r}$. The proper Delsarte type operators $\mathbf{\Omega}_{\pm} : \mathcal{H}^0_{\Lambda(\mathcal{L}),-}(M) \to \mathcal{H}^0_{\Lambda(\tilde{\mathcal{L}}),-}(M)$ are related with two different realizations of the action (5.28) under the necessary conditions

$$d_{\tilde{\mathcal{L}}} \tilde{\psi}^{(0)}(\eta)dx = 0, \quad d^*_{\tilde{\mathcal{L}}} \tilde{\varphi}^{(0)}(\lambda) = 0, \tag{5.36}$$

needed to be satisfied and meaning, evidently, that the embeddings $\tilde{\varphi}^{(0)}(\lambda) \in \mathcal{H}^0_{\Lambda(\tilde{\mathcal{L}}^*),-}(M)$, $\lambda \in \Sigma$, and $\tilde{\psi}^{(0)}(\eta)dx \in \mathcal{H}^s_{\Lambda(\tilde{\mathcal{L}}),-}(M)$, $\eta \in \Sigma$, are satisfied. Now we need to formulate a lemma being important for the conditions (5.36) to hold.

Lemma 5.6. *The following invariance property*

$$\tilde{Z}^{(s)} = \Omega_{(t_0;x_0)} \Omega^{-1}_{(t;x)} Z^{(s)} \Omega^{-1}_{(t;x)} \Omega_{(t_0;x_0)} \tag{5.37}$$

holds for any $(t;x)$ and $(t_0;x_0) \in M$.

As a result of (5.37) and the symmetry invariance between cohomology spaces $\mathcal{H}^0_{\Lambda(\mathcal{L}),-}(M)$ and $\mathcal{H}^0_{\Lambda(\tilde{\mathcal{L}}),-}(M)$ one obtains the following pairs of related mappings:

$$\begin{aligned} \psi^{(0)} &= \tilde{\psi}^{(0)} \tilde{\Omega}^{-1}_{(t;x)} \tilde{\Omega}_{(t_0;x_0)}, \quad \varphi^{(0)} = \tilde{\varphi}^{(0)} \tilde{\Omega}^{\circledast,-1}_{(t;x)} \tilde{\Omega}^{\circledast}_{(t_0;x_0)}, \\ \tilde{\psi}^{(0)} &= \psi^{(0)} \Omega^{-1}_{(t;x)} \Omega_{(t_0;x_0)}, \quad \tilde{\varphi}^{(0)} = \varphi^{(0)} \Omega^{\circledast,-1}_{(t;x)} \Omega^{\circledast}_{(t_0;x_0)}, \end{aligned} \tag{5.38}$$

where the integral operator kernels from $L^{(\rho)}_2(\Sigma;\mathbb{C}) \otimes L^{(\rho)}_2(\Sigma;\mathbb{C})$ are defined as

$$\begin{aligned} \tilde{\Omega}_{(t;x)}(\lambda,\mu) &:= \int_{\sigma^{(s)}_{(t;x)}} \tilde{\Omega}^{(s-2)}[\tilde{\varphi}^{(0)}(\lambda), \tilde{\psi}^{(0)}(\mu)dx], \\ \tilde{\Omega}^{\circledast}_{(t;x)}(\lambda,\mu) &:= \int_{\sigma^{(s)}_{(t;x)}} \bar{\tilde{\Omega}}^{(s-2),\intercal}[\tilde{\varphi}^{(0)}(\lambda), \tilde{\psi}^{(0)}(\mu)dx] \end{aligned} \tag{5.39}$$

for all $(\lambda,\mu) \in \Sigma \times \Sigma$, giving rise to finding proper Delsarte transmutation operators ensuring the pure differential nature of the transformed expressions (5.32).

Note here also that due to (5.37) and (5.38) the following operator property

$$\Omega_{(t_0;x_0)} \Omega^{-1}_{(t;x)} \Omega_{(t_0;x_0)} + \tilde{\Omega}_{(t_0;x_0)} \Omega^{-1}_{(t;x)} \Omega_{(t_0;x_0)} = 0 \tag{5.40}$$

holds for any $(t_0;x_0)$ and $(t;x)\in M$ meaning that $\tilde{\Omega}_{(t_0;x_0)}=-\Omega_{(t_0;x_0)}$.

5.6. One can now define similar to (5.16) the additional closed and dense in $\mathcal{H}^0_{\Lambda,-}(M)$ three subspaces

$$\mathcal{H}_0:=\{\psi^{(0)}(\mu)\in\mathcal{H}^0_{\Lambda(\mathcal{L}),-}(M):d_{\mathcal{L}}\psi^{(0)}(\mu)=0,\quad \psi^{(0)}(\mu)|_{\Gamma}=0,\ \mu\in\Sigma\},$$

$$\tilde{\mathcal{H}}_0:=\{\tilde{\psi}^{(0)}(\mu)\in\mathcal{H}^0_{\Lambda(\tilde{\mathcal{L}}),-}(M):d_{\tilde{\mathcal{L}}}\tilde{\psi}^{(0)}(\mu)=0,\quad \tilde{\psi}^{(0)}(\mu)|_{\tilde{\Gamma}}=0,\ \mu\in\Sigma\},\tag{5.41}$$

$$\tilde{\mathcal{H}}^*_0:=\{\tilde{\varphi}^{(0)}(\eta)\in\mathcal{H}^0_{\Lambda(\tilde{\mathcal{L}}^*),-}(M):d^*_{\tilde{\mathcal{L}}}\tilde{\psi}^{(0)}(\eta)=0,\quad \tilde{\varphi}^{(0)}(\eta)|_{\tilde{\Gamma}}=0,\ \eta\in\Sigma\},$$

where Γ and $\tilde{\Gamma}\subset M$ are some smooth $(s-1)$-dimensional hypersurfaces, and construct the actions

$$\mathbf{\Omega}_{\pm}:\psi^{(0)}\to\tilde{\psi}^{(0)}:=\psi^{(0)}\Omega^{-1}_{(t;x)}\Omega_{(t;x)},\qquad \mathbf{\Omega}^{\circledast}_{\pm}:\varphi^{(0)}\to\tilde{\varphi}^{(0)}:=\varphi^{(0)}\Omega^{\circledast,-1}_{(t;x)}\Omega^{\circledast}_{(t_0;x_0)}\tag{5.42}$$

on arbitrary but fixed pairs of elements $(\varphi^{(0)}(\lambda),\psi^{(0}(\mu))\in\mathcal{H}^*_0\times\mathcal{H}_0$, parametrized by the set Σ, where by definition, one needs that all obtained pairs $(\tilde{\varphi}^{(0)}(\lambda),\tilde{\psi}^{(0)}(\mu)dx)$, $\lambda,\mu\in\Sigma$, belong to $\mathcal{H}^0_{\Lambda(\tilde{\mathcal{L}}^*),-}(M)\times\mathcal{H}^s_{\Lambda(\tilde{\mathcal{L}}),-}(M)$. Note also that related operator property (5.40) can be compactly written down as follows:

$$\tilde{\Omega}_{(t;x)}=\tilde{\Omega}_{(t_0;x_0)}\Omega^{-1}_{(t;x)}\Omega_{(t_0;x_0)}=-\Omega_{(t_0;x_0)}\Omega^{-1}_{(t;x)}\Omega_{(t_0;x_0)}.\tag{5.43}$$

Construct now from the expressions (5.42) the following operator kernels from the Hilbert space $L^{(\rho)}_2(\Sigma;\mathbb{C})\otimes L^{(\rho)}_2(\Sigma;\mathbb{C})$:

$$\begin{aligned}\Omega_{(t;x)}(\lambda,\mu)-\Omega_{(t_0;x_0)}(\lambda,\mu)&=\int_{\partial S^{(s)}_{(t;x)}}\Omega^{(s-1)}[\varphi^{(0)}(\lambda),\psi^{(0)}(\mu)dx]\\ :\quad -\int_{\partial S^{(s)}_{(t_0;x_0)}}\Omega^{(s-1)}[\varphi^{(0)}(\lambda),\psi^{(0)}(\mu)dx]&\\ =\int\limits_{S^{(s)}_{\pm}(\sigma^{(s-1)}_{(t;x)},\sigma^{(s-1)}_{(t_0;x_0)})}d\Omega^{(s-1)}[\varphi^{(0)}(\lambda),\psi^{(0)}(\mu)dx]&\\ =\int\limits_{S^{(s)}_{\pm}(\sigma^{(s-1)}_{(t;x)},\sigma^{(s-1)}_{(t_0;x_0)})}Z^{(s)}[\varphi^{(0)}(\lambda),\psi^{(0)}(\mu)dx],&\end{aligned}\tag{5.44}$$

and, similarly,

$$\begin{aligned}\Omega^{\circledast}_{(t;x)}(\lambda,\mu)-\Omega^{\circledast}_{(t_0;x_0)}(\lambda,\mu)&=\int_{\partial S^{(s)}_{(t;x)}}\bar{\Omega}^{(s-1),\intercal}[\varphi^{(0)}(\lambda),\psi^{(0)}(\mu)dx]\\ -\int_{\partial S^{(s)}_{(t_0;x_0)}}\bar{\Omega}^{(s-1),\intercal}[\varphi^{(0)}(\lambda),\psi^{(0)}(\mu)dx]&\end{aligned}\tag{5.45}$$

$$= \int\limits_{S_{\pm}^{(s)}(\sigma_{(t;x)}^{(s-1)},\sigma_{(t_0;x_0)}^{(s-1)})} d\bar{\Omega}^{(s-1),\intercal}[\varphi^{(0)}(\lambda),\psi^{(0)}(\mu)dx]$$

$$= \int\limits_{S_{\pm}^{(s)}(\sigma_{(t;x)}^{(s-1)},\sigma_{(t_0;x_0)}^{(s-1)})} \bar{Z}^{(s-1),\intercal}[\varphi^{(0)}(\lambda),\psi^{(0)}(\mu)dx],$$

where $\lambda,\mu\in\Sigma$, and by definition, s-dimensional surfaces $S_{+}^{(s)}(\sigma_{(t;x)}^{(s-1)},\sigma_{(t_0;x_0)}^{(s-1)}))$ and $S_{-}^{(s)}(\sigma_{(t;x)}^{(s-1)},\sigma_{(t_0;x_0)}^{(s-1)})\subset M$ are spanned smoothly without self-intersection between two homological cycles $\sigma_{(t;x)}^{(s-1)}=\partial S_{(t;x)}^{(s)}$ and $\sigma_{(t_0;x_0)}^{(s-1)}:=\partial S_{(t_0;x_0)}^{(s)}\in C_{s-1}(M;\mathbb{C})$ in such a way that the boundary $\partial(S_{+}^{(s)}(\sigma_{(t;x)}^{(s-1)},\sigma_{(t_0;x_0)}^{(s-1)})\cup S_{-}^{(s)}(\sigma_{(t;x)}^{(s-1)},\sigma_{(t_0;x_0)}^{(s-1)}))=\oslash$. Since the integral operator expressions $\Omega_{(t_0;x_0)},\Omega_{(t_0;x_0)}^{\circledast}:L_2^{(\rho)}(\Sigma;\mathbb{C})\to L_2^{(\rho)}(\Sigma;\mathbb{C})$ are at a fixed point $(t_0;x_0)\in M$, evidently, constant and assumed to be invertible, for extending the actions given (5.42) on the whole Hilbert space $\mathcal{H}\times\mathcal{H}^*$ one can apply to them the classical constants variation approach, making use of the expressions (5.45). As a result, we obtain easily the following Delsarte transmutation integral operator expressions

(5.46)

$$\mathbf{\Omega}_{\pm}=\mathbf{1}-\int_{\Sigma\times\Sigma}d\rho(\xi)d\rho(\eta)\tilde{\psi}(x;\xi)\Omega_{(t_0;x_0)}^{-1}(\xi,\eta)\int\limits_{S_{\pm}^{(s)}(\sigma_{(t;x)}^{(s-1)},\sigma_{(t_0;x_0)}^{(s-1)})}Z^{(s)}[\varphi^{(0)}(\eta),\cdot],$$

$$\mathbf{\Omega}_{\pm}^{\circledast}=\mathbf{1}-\int_{\Sigma\times\Sigma}d\rho(\xi)d\rho(\eta)\tilde{\varphi}(x;\eta)\Omega_{(t_0;x_0)}^{\circledast,-1}(\xi,\eta)\int\limits_{S_{\pm}^{(s)}(\sigma_{(t;x)}^{(s-1)},\sigma_{(t_0;x_0)}^{(s-1)})}\bar{Z}^{(s),\intercal}[\cdot,\psi^{(0)}(\xi)dx]$$

for fixed pairs $(\varphi^{(0)}(\xi),\psi^{(0)}(\eta))\in\mathcal{H}_0^*\times\mathcal{H}_0$ and $(\tilde{\varphi}^{(0)}(\lambda),\tilde{\psi}^{(0)}(\mu))\in\tilde{\mathcal{H}}_0^*\times\tilde{\mathcal{H}}_0$, $\lambda,\mu\in\Sigma$, being bounded invertible integral operators of Volterra type on the whole space $\mathcal{H}\times\mathcal{H}^*$. Applying the same arguments as in Section 1, one can show also that correspondingly transformed sets of operators $\tilde{\mathrm{L}}_j:=\mathbf{\Omega}_{\pm}\mathrm{L}_j\mathbf{\Omega}_{\pm}^{-1}$, $j=\overline{1,r}$, and $\tilde{\mathrm{L}}_k^*:=\mathbf{\Omega}_{\pm}^{\circledast}\mathrm{L}_k^*\mathbf{\Omega}_{\pm}^{\circledast,-1}$, $k=\overline{1,r}$, prove to be purely differential too. Thereby, one can formulate the following final theorem.

Theorem 5.7. *The expressions (5.46) are bounded invertible Delsarte transmutation integral operators of Volterra type onto $\mathcal{H}\times\mathcal{H}^*$, transforming, correspondingly, given commuting sets of operators L_j, $j=\overline{1,r}$, and their formally adjoint ones L_k^*, $k=\overline{1,r}$, into the pure differential sets of operators $\tilde{\mathrm{L}}_j:=\mathbf{\Omega}_{\pm}\mathrm{L}_j\mathbf{\Omega}_{\pm}^{-1}$, $j=\overline{1,r}$, and $\tilde{\mathrm{L}}_k^*:=\mathbf{\Omega}_{\pm}^{\circledast}\mathrm{L}_k^*\mathbf{\Omega}_{\pm}^{\circledast,-1}$, $k=\overline{1,r}$. Moreover, the suitably constructed closed subspaces $\mathcal{H}_0\subset\mathcal{H}$ and $\tilde{\mathcal{H}}_0\subset\mathcal{H}$, such that the operator $\mathbf{\Omega}\in Aut(\mathcal{H})\cap\mathcal{B}(\mathcal{H})$ depend strongly on the topological structure of the generalized cohomology groups $\mathcal{H}_{\Lambda(\mathcal{L}),-}^0(M)$ and $\mathcal{H}_{\Lambda(\tilde{\mathcal{L}}),-}^0(M)$, being parametrized by elements $S_{\pm}^{(s)}(\sigma_{(t;x)}^{(s-1)},\sigma_{(t_0;x_0)}^{(s-1)})\in H_s(M;\mathbb{C})$.*

5.7. Suppose now that all of differential operators $L_j:=L_j(x|\partial)$, $j=\overline{1,r}$, considered above don't depend on the variable $t\in T^r\subset\mathbb{R}_+^r$. Then, evidently, one can take

$$\begin{aligned}\mathcal{H}_0\quad &:\quad =\{\psi_\mu^{(0)}(\xi)\in L_{2,-}(\mathbb{R}^s;\mathbb{C}^N):L_j\psi_\mu^{(0)}(\xi)=\mu_j\psi_\mu^{(0)}(\xi),\quad j=\overline{1,r},\\ \psi_\mu^{(0)}(\xi)|_\Gamma\quad &=\quad 0,\ \mu:=(\mu_1,...,\mu_r)\in\sigma(\tilde{L})\cap\bar{\sigma}(L^*),\ \xi\in\Sigma_\sigma\},\end{aligned}$$

$$\begin{aligned}
\tilde{\mathcal{H}}_0 &: = \{\tilde{\psi}_\mu^{(0)}(\xi) \in L_{2,-}(\mathbb{R}^s;\mathbb{C}^N) : \tilde{L}_j \tilde{\psi}_\mu^{(0)}(\xi) = \mu_j \tilde{\psi}_\mu^{(0)}(\xi), \quad j=\overline{1,r}, \\
\tilde{\psi}_\mu^{(0)}(\xi)|_{\tilde{\Gamma}} &= 0, \ \mu := (\mu_1,...,\mu_r) \in \sigma(\tilde{L}) \cap \bar{\sigma}(L^*), \ \xi \in \Sigma_\sigma\},
\end{aligned}$$

$$\begin{aligned}
\mathcal{H}_0^* &: = \{\varphi_\lambda^{(0)}(\eta) \in L_{2,-}(\mathbb{R}^s;\mathbb{C}^N) : L_j \varphi_\lambda^{(0)}(\eta) = \bar{\lambda}_j \varphi_\lambda^{(0)}(\eta), \quad j=\overline{1,r}, \\
\varphi_\lambda^{(0)}(\eta)|_{\Gamma} &= 0, \ \lambda := (\lambda_1,...,\lambda_r) \in \sigma(\tilde{L}) \cap \bar{\sigma}(L^*), \ \eta \in \Sigma_\sigma\},
\end{aligned} \tag{5.47}$$

$$\begin{aligned}
\tilde{\mathcal{H}}_0^* &: = \{\tilde{\varphi}_\lambda^{(0)}(\eta) \in L_{2,-}(\mathbb{R}^s;\mathbb{C}^N) : \tilde{L}_j \tilde{\varphi}_\lambda^{(0)}(\eta) = \bar{\lambda}_j \tilde{\varphi}_\lambda^{(0)}(\eta), \quad j=\overline{1,r}, \\
\tilde{\varphi}_\lambda^{(0)}(\eta)|_{\tilde{\Gamma}} &= 0, \ \lambda := (\lambda_1,...,\lambda_r) \in \sigma(\tilde{L}) \cap \bar{\sigma}(L^*), \ \eta \in \Sigma_\sigma\}
\end{aligned}$$

and construct the corresponding Delsarte transmutation operators

$$\begin{aligned}
\mathbf{\Omega}_\pm &= \mathbf{1} - \int\limits_{\sigma(\tilde{L})\cap\bar{\sigma}(L^*)} d\rho_\sigma(\lambda) \int\limits_{\Sigma_\sigma\times\Sigma_\sigma} d\rho_{\Sigma_\sigma}(\xi) d\rho_{\Sigma_\sigma}(\eta) \\
&\quad \times \int\limits_{S_\pm^{(s)}(\sigma_x^{(s-1)},\sigma_{x_0}^{(s-1)})} dx \quad \tilde{\psi}_\lambda^{(0)}(\xi) \Omega_{(x_0)}^{-1}(\lambda;\xi,\eta) \bar{\tilde{\varphi}}_\lambda^{(0),\intercal}(\eta)(\cdot)
\end{aligned} \tag{5.48}$$

and

$$\begin{aligned}
\mathbf{\Omega}_\pm^{\circledast} &= \mathbf{1} - \int\limits_{\sigma(\tilde{L})\cap\bar{\sigma}(L^*)} d\rho_\sigma(\lambda) \int_{\Sigma} d\rho_{\Sigma_\sigma}(\xi) d\rho_{\Sigma_\sigma}(\eta) \\
&\quad \times \int\limits_{S_\pm^{(s)}(\sigma_x^{(s-1)},\sigma_{x_0}^{(s-1)})} dx \quad \tilde{\varphi}_\lambda^{(0)}(\xi) \bar{\Omega}_{(x_0)}^{\intercal,-1}(\lambda;\xi,\eta) \times \bar{\tilde{\psi}}_\lambda^{(0),\intercal}(\eta)(\cdot),
\end{aligned} \tag{5.49}$$

acting already in the Hilbert space $L_2(\mathbb{R}^s;\mathbb{C}^N)$, where for any $(\lambda;\xi,\eta) \in (\sigma(\tilde{L}) \cap \bar{\sigma}(L^*)) \times \Sigma_\sigma^2$ kernels

$$\begin{aligned}
\Omega_{(x_0)}(\lambda;\xi,\eta) &: = \int_{\sigma_{x_0}^{(s-1)}} \Omega^{(s-1)}[\varphi_\lambda^{(0)}(\xi), \psi_\lambda^{(0)}(\eta) dx], \\
\Omega_{(x_0)}^{\circledast}(\lambda;\xi,\eta) &: = \int_{\sigma_{x_0}^{(s-1)}} \bar{\Omega}^{(s-1),\intercal}[\varphi_\lambda^{(0)}(\xi), \psi_\lambda^{(0)}(\eta) dx]
\end{aligned} \tag{5.50}$$

belong to $L_2^{(\rho)}(\Sigma_\sigma;\mathbb{C}) \times L_2^{(\rho)}(\Sigma_\sigma;\mathbb{C})$ for every $\lambda \in \sigma(\tilde{L}) \cap \bar{\sigma}(L^*)$ considered as a parameter. Moreover, as $\partial \mathbf{\Omega}_\pm / \partial t_j = 0$, $j=\overline{1,r}$, one gets easily the set of differential expressions

$$\mathcal{R}(\tilde{L}) := \{\tilde{L}_j(x|\partial) := \mathbf{\Omega}_\pm L_j(x|\partial) \mathbf{\Omega}_\pm^{-1} : j=\overline{1,r}\}, \tag{5.51}$$

being a ring of commuting to each other differential operators acting in $L_2(\mathbb{R}^s;\mathbb{C}^N)$, generated by the corresponding initial ring $\mathcal{R}(L)$.

Thus we have described above a ring $\mathcal{R}(\tilde{L})$ of commuting to each other multidimensional differential operators, generated by an initial ring $\mathcal{R}(L)$. This problem in the one-dimensional case was before treated in detail and effectively solved in [6, 16] by means of algebro-geometric and inverse spectral transform techniques. Our approach gives another look at this problem in multidimension and is of special interest due to its clear and readable dependence on dimension of differential operators.

6. A special case: soliton theory aspect

6.1. Consider our de Rham-Hodge theory of a commuting set $\mathcal{L}$ of two differential operators in a Hilbert space $\mathcal{H} := L_2(\mathrm{T}^2; H)$, $H := L_2(\mathbb{R}^s; \mathbb{C}^N)$, for the special case when $M := \mathrm{T}^2 \times \bar{\mathbb{R}}^s$ and

$$\mathcal{L} := \{\mathrm{L}_j := \partial/\partial t_j - L_j(t; x|\partial) : t_j \in \ \mathrm{T}_j := [0, T_j) \subset \mathbb{R}_+, \ j = \overline{1,2}\},$$

where, by definition, $\mathrm{T}^2 := \mathrm{T}_1 \times \mathrm{T}_2$,

$$L_j(t; x|\partial) := \sum_{|\alpha|=0}^{n(L_j)} a_\alpha^{(j)}(t; x)\partial^{|\alpha|}/\partial x^\alpha \tag{6.1}$$

with coefficients $a_\alpha^{(j)} \in C^1(\mathrm{T}^2; S(\mathbb{R}^s; End\mathbb{C}^N))$, $\alpha \in \mathbb{Z}_+^s$, $|\alpha| = \overline{0, n(L_j)}$, $j = \overline{1,2}$. The corresponding scalar product is given now as

$$(\varphi, \psi) := \int_{\mathrm{T}^2} dt \int_{\mathbb{R}^s} dx < \varphi, \psi > \tag{6.2}$$

for any pair $(\varphi, \psi) \in \mathcal{H} \times \mathcal{H}$ and the generalized external differential

$$d_{\mathcal{L}} := \sum_{j=1}^{2} dt_j \wedge \mathrm{L}_j, \tag{6.3}$$

where one assumes that for all $t \in \mathrm{T}^2$ and $x \in \mathbb{R}^s$ the commutator

$$[\mathrm{L}_1, \mathrm{L}_2] = 0. \tag{6.4}$$

Tis means, obviously, that the corresponding de Rham-Hodge-Skrypnik co-chain complexes

$$\begin{aligned} \mathcal{H} \quad &\rightarrow \quad \Lambda^0(M; \mathcal{H}) \overset{d_{\mathcal{L}}}{\rightarrow} \Lambda^1(M; \mathcal{H}) \overset{d_{\mathcal{L}}}{\rightarrow} ... \overset{d_{\mathcal{L}}}{\rightarrow} \Lambda^m(M; \mathcal{H}) \overset{d_{\mathcal{L}}}{\rightarrow} 0, \\ \mathcal{H} \quad &\rightarrow \quad \Lambda^0(M; \mathcal{H}) \overset{d_{\mathcal{L}}^*}{\rightarrow} \Lambda^1(M; \mathcal{H}) \overset{d_{\mathcal{L}}^*}{\rightarrow} ... \overset{d_{\mathcal{L}}^*}{\rightarrow} \Lambda^m(M; \mathcal{H}) \overset{d_{\mathcal{L}}^*}{\rightarrow} 0 \end{aligned} \tag{6.5}$$

are exact. Define now due to (5.16) and (5.41) the closed subspaces $H_0^{\circledast}$ and $H_0 \subset H_-$ as follows:

$$\begin{aligned} \mathcal{H}_0 \quad &: \quad = \{\psi^{(0)}(\lambda; \eta) \in \mathcal{H}^0_{\Lambda(\mathcal{L}),-}(M) : \\ \partial\psi^{(0)}(\lambda; \eta)/\partial t_j \quad &= \quad L_j(t; x|\partial)\psi^{(0)}(\lambda; \eta), \ j = \overline{1,2}, \\ \psi^{(0)}(\lambda; \eta)|_{t=t_0} \quad &= \quad \psi_\lambda(\eta) \in H_-, \ \psi^{(0)}(\lambda; \eta)|_\Gamma = 0, \\ (\lambda; \eta) \quad &\in \quad \Sigma \subset (\sigma(L) \cap \bar{\sigma}(L^*)) \times \Sigma_\sigma\}, \end{aligned} \tag{6.6}$$

$$\begin{aligned} \mathcal{H}_0^* \quad &: \quad = \{\varphi^{(0)}(\lambda; \eta) \in \mathcal{H}^0_{\Lambda(\mathcal{L}),-}(M) : \\ -\partial\varphi^{(0)}(\lambda; \eta)/\partial t_j \quad &= \quad L_j(t; x|\partial)\varphi^{(0)}(\lambda; \eta), \ j = \overline{1,2}, \\ \varphi^{(0)}(\lambda; \eta)|_{t=t_0} \quad &= \quad \varphi_\lambda(\eta) \in H_-, \ \varphi^{(0)}(\lambda; \eta)|_\Gamma = 0, \\ (\lambda; \eta) \quad &\in \quad \Sigma \subset (\sigma(L) \cap \bar{\sigma}(L^*)) \times \Sigma_\sigma\} \end{aligned}$$

for some hypersurface $\Gamma \subset M$ and a "spectral" degeneration set $\Sigma_\sigma \in \mathbb{C}^{p-1}$. By means of subspaces (6.6) one can now proceed to construction of Delsarte transmutation operators $\mathbf{\Omega}_\pm : H \leftrightarrows H$ in the general form like (5.49) with kernels

$\Omega_{(t_0;x_0)}(\lambda;\xi,\eta) \in L_2^{(\rho)}(\Sigma_\sigma;\mathbb{C}) \times L_2^{(\rho)}(\Sigma_\sigma;\mathbb{C})$ for every $\lambda \in \sigma(L) \cap \bar{\sigma}(L^*)$, being defined as

$$\begin{aligned}\Omega_{(t_0;x_0)}(\lambda;\xi,\eta) \ &:= \int_{\sigma^{(s-1)}_{(t_0;x_0)}} \Omega^{(s-1)}[\varphi^{(0)}(\lambda;\xi),\psi^{(0)}(\lambda;\eta)dx], \\ \Omega^{\circledast}_{(t_0;x_0)}(\lambda;\xi,\eta) \ &:= \int_{\sigma^{(s-1)}_{(t_0;x_0)}} \bar{\Omega}^{(s-1),\intercal}[\varphi^{(0)}(\lambda;\xi),\psi^{(0)}(\lambda;\eta)dx]\end{aligned} \tag{6.7}$$

for all $(\lambda;\xi,\eta) \in (\sigma(L) \cap \bar{\sigma}(L^*)) \times \Sigma_\sigma^2$. As a result one gets for the corresponding product $\rho := \rho_\sigma \odot \rho_{\Sigma_\sigma^2}$ such integral expressions:

$$\begin{aligned}\mathbf{\Omega}_\pm \ &= \ \mathbf{1} - \int\limits_{\sigma(\tilde{L})\cap\bar{\sigma}(L^*)} d\rho_\sigma(\lambda) \int\limits_{\Sigma_\sigma\times\Sigma_\sigma} d\rho_{\Sigma_\sigma}(\xi)d\rho_{\Sigma_\sigma}(\eta) \\ &\times \int\limits_{S_\pm^{(s)}(\sigma^{(s-1)}_{(t_0;x)},\sigma^{(s-1)}_{(t_0;x_0)})} dx \qquad \tilde{\psi}^{(0)}(\lambda;\xi)\Omega^{-1}_{(t_0;x_0)}(\lambda;\xi,\eta)\bar{\varphi}^{(0),\intercal}(\lambda;\eta)(\cdot),\end{aligned}$$

$$\begin{aligned}\mathbf{\Omega}^{\circledast}_\pm \ &= \ \mathbf{1} - \int\limits_{\sigma(\tilde{L})\cap\bar{\sigma}(L^*)} d\rho_\sigma(\lambda) \int\limits_{\Sigma_\sigma\times\Sigma_\sigma} d\rho_{\Sigma_\sigma}(\xi)d\rho_{\Sigma_\sigma}(\eta) \\ &\times \int\limits_{S_\pm^{(s)}(\sigma^{(s-1)}_{(t_0;x)},\sigma^{(s-1)}_{(t_0;x_0)})} dx \qquad \tilde{\varphi}^{(0)}_\lambda(\xi)\bar{\Omega}^{\intercal,-1}_{(t_0;x_0)}(\lambda;\xi,\eta) \times \bar{\psi}^{(0),\intercal}(\lambda;\eta)(\cdot),\end{aligned} \tag{6.8}$$

where $S_+^{(s)}(\sigma^{(s-1)}_{(t_0;x)},\sigma^{(s-1)}_{(t_0;x_0)}) \in H_s(M;\mathbb{C})$ is some smooth s-dimensional surface between two homological cycles $\sigma^{(s-1)}_{(t_0;x)}$ and $\sigma^{(s-1)}_{(t_0;x_0)} \in \mathcal{K}(M)$ and $S_-^{(s)}(\sigma^{(s-1)}_{(t_0;x)},\sigma^{(s-1)}_{(t_0;x_0)}) \in H_s(M;\mathbb{C})$ is its smooth counterpart such that $\partial(S_+^{(s)}(\sigma^{(s-1)}_{(t_0;x)},\sigma^{(s-1)}_{(t_0;x_0)}) \cup S_-^{(s)}(\sigma^{(s-1)}_{(t_0;x)},\sigma^{(s-1)}_{(t_0;x_0)})) = \oslash$. Concerning the related results of Chapter 3 one can construct from (6.8) the corresponding factorized Fredholm operators $\mathbf{\Omega}$ and $\mathbf{\Omega}^{\circledast} : H \to H$, $H = L_2(\mathbb{R};\mathbb{C}^N)$, as follows:

$$\mathbf{\Omega} := \mathbf{\Omega}_+^{-1}\mathbf{\Omega}_-, \qquad \mathbf{\Omega}^{\circledast} := \mathbf{\Omega}_+^{\circledast-1}\mathbf{\Omega}_-^{\circledast}. \tag{6.9}$$

It is also important to notice here that kernels $\hat{K}_\pm(\mathbf{\Omega})$ and $\hat{K}_\pm(\mathbf{\Omega}^{\circledast}) \in H_- \otimes H_-$ satisfy exactly the generalized [8] determining equations in the following tensor form

$$\begin{aligned}(\tilde{\mathcal{L}} \otimes \mathbf{1})\hat{K}_\pm(\mathbf{\Omega}) \ &= \ (\mathbf{1}\otimes\mathcal{L}^*)\hat{K}_\pm(\mathbf{\Omega}), \\ (\tilde{\mathcal{L}}^* \otimes \mathbf{1})\hat{K}_\pm(\mathbf{\Omega}^{\circledast}) \ &= \ (\mathbf{1}\otimes\mathrm{L})\hat{K}_\pm(\mathbf{\Omega}^{\circledast}).\end{aligned} \tag{6.10}$$

Since, evidently, $supp\hat{K}_+(\mathbf{\Omega}) \cap supp\hat{K}_-(\mathbf{\Omega}) = \oslash$ and $supp\hat{K}_+(\mathbf{\Omega}^{\circledast}) \cap supp\hat{K}_-(\mathbf{\Omega}^{\circledast}) = \oslash$, one derives from results [19, 22, 21] that corresponding Gelfand-Levitan-Marchenko equations

$$\begin{aligned}\hat{K}_+(\mathbf{\Omega}) + \hat{\Phi}(\mathbf{\Omega}) + \hat{K}_+(\mathbf{\Omega}) * \hat{\Phi}(\mathbf{\Omega}) \ &= \ \hat{K}_-(\mathbf{\Omega}), \\ \hat{K}_+(\mathbf{\Omega}^{\circledast}) + \hat{\Phi}(\mathbf{\Omega}^{\circledast}) + \hat{K}_+(\mathbf{\Omega}^{\circledast}) * \hat{\Phi}(\mathbf{\Omega}^{\circledast}) \ &= \ \hat{K}_-(\mathbf{\Omega}^{\circledast}),\end{aligned} \tag{6.11}$$

where, by definition, $\mathbf{\Omega} := \mathbf{1} + \hat{\Phi}(\mathbf{\Omega})$, $\mathbf{\Omega}^{\circledast} := 1 + \hat{\Phi}(\mathbf{\Omega}^{\circledast})$, can be solved [19, 18] in the space $\mathcal{B}^\pm_\infty(H)$ for kernels $\hat{K}_\pm(\mathbf{\Omega})$ and $\hat{K}_\pm(\mathbf{\Omega}^{\circledast}) \in H_-\otimes H_-$ depending parametrically on $t \in \mathrm{T}^2$. Thereby, Delsarte transformed differential operators $\tilde{\mathrm{L}}_j : \mathcal{H} \to \mathcal{H}$,

$j=\overline{1,2}$, will be, evidently, commuting to each other too, satisfying the following operator relationships:

$$\tilde{\mathrm{L}}_j = \partial/\partial t_j - \mathbf{\Omega}_{\pm} L_j \mathbf{\Omega}_{\pm}^{-1} - (\partial \mathbf{\Omega}_{\pm}/\partial t_j)\mathbf{\Omega}_{\pm}^{-1} := \partial/\partial t_j - \tilde{L}_j, \tag{6.12}$$

where operator expressions for $\tilde{L}_j : H \to H$, $j=\overline{1,2}$, prove to be purely differential. The latter property makes it possible to construct some nonlinear in general partial differential equations on coefficients of differential operators (6.12) and solve them by means of the standard procedures either of inverse spectral [6, 11, 13, 7] or the Darboux-Backlund [14, 20, 24] transforms, producing a wide class of exact soliton like solutions. Another not simple and very interesting aspect of the approach devised in this paper concerns regular algorithms of treating differential operator expressions depending on a "spectral" parameter $\lambda \in \mathbb{C}$, which was just a little recently discussed in [22, 21].

7. Conclusion

The results obtained above and developing the De Rham-Hodge-Skrypnik theory [28, 29, 27, 3, 30] of special differential complexes gave rise to effective analytical expressions for the corresponding Delsarte transmutation Volterra type operators in a given Hilbert space $\mathcal{H}$. In particular, it was shown also that they can be effectively applied to studying the integral operator structure of Delsarte transmutation operators for polynomial pencils of differential operators in $\mathcal{H}$ having many applications both in spectral theory of such multidimensional operator pencils and in soliton theory [17, 12, 6, 22, 25] of multidimensional integrable dynamical systems on functional manifolds, being very important for diverse applications in modern mathematical physics. If one considers a differential operator $L:\mathcal{H}\to\mathcal{H}$ and assumes that its spectrum $\sigma(L)$ consists of the discrete $\sigma_d(L)$ and continuos $\sigma_c(L)$ parts, by means of the general form of the Delsarte transmutation operators obtained in Chapters 4 and 5 one can construct a new more complicated differential operator $\tilde{L} := \mathbf{\Omega}_{\pm} L \mathbf{\Omega}_{\pm}^{-1}$ in $\mathcal{H}$, such that its continuous spectrum $\sigma_c(\tilde{L}) = \sigma_c(L)$ but $\sigma_d(L) \neq \sigma_d(\tilde{L})$. Thereby these Delsarte transformed operators can be effectively used for both studying generalized spectral properties of differential operators and operator pencils [8, 5, 11, 13, 9, 32, 31] and constructing a wide class of nontrivial differential operators with a prescribed spectrum as it was done [11, 6, 33] in one dimension.

As it was shown before in [5, 17] for the two-dimensional Dirac and three-dimensional perturbed Laplace operators, the kernels of the corresponding Delsarte transmutation operator satisfy some special of Fredholm type linear integral equations called the Gelfand-Levitan-Marchenko ones, which are of very importance for solving the corresponding inverse spectral problem and having many applications in modern mathematical physics. Such equations can be easily constructed for our multidimensional case too, thereby making it possible to pose the corresponding inverse spectral problem for describing a wide class of multidimensional operators with a priori given spectral characteristics. Also, similar to [17, 25, 6, 11], one can use such results for studying so called completely integrable nonlinear evolution equations, especially for constructing by means of special Darboux type transformations [14, 20, 24] their exact solutions like solitons and many others. Such an activity is now in progress and the corresponding results will be published later.

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